Willmore surfaces in S^{n+2} by the loop group method: generic cases and some examples

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Abstract

In this paper we deal with the global properties of Willmore surfaces in spheres via the harmonic conformal Gauss map using loop groups.

A main result is a global description of the harmonic maps corresponding to Willmore surfaces (Theorem 3.10 and Theorem 3.16).

As a second main result, for the construction of examples, we identify specific types of potentials (for the loop group formalism) which are characteristic for certain types of Willmore surfaces, like Willmore spheres, equivariant Willmore surfaces or Willmore surfaces with finite order rotational symmetries.

For the case of Willmore surfaces we prove that every (conformally) harmonic map into some non-compact symmetric space G/K induces a (conformally) harmonic map into the compact dual space $U/(U \cap K^{\mathbb{C}})$. As a consequence all Willmore spheres are of finite uniton type.

The third main result is the construction of new examples, in particular of an explicit, unbranched (isotropic) Willmore sphere into S^6 which is not S-Willmore.

Keywords: Willmore surfaces; conformal Gauss maps; normalized potential; non-compact symmetric space; Iwasawa decompositions.

1 Introduction

Immersions which are critical points of certain functionals have been investigated from the beginning of differential geometry. It is surprising that the "total mean curvature functional"

$$\tilde{\mathcal{W}} = \int_M H^2 dM$$

for an immersion did not receive much attention until 1965, when Willmore stated his famous conjecture [62] on the Willmore functional of 2-tori. This seems to be the more surprising since the total Gauss curvature integral $\int_M K dM$ of a closed surface M, relating the topology to metric properties of a surface via the celebrated Gauss-Bonnet formula, is frequently used.

It was realized soon that the "Willmore functional"

$$\mathcal{W} = \int_M (H^2 - K) dM$$

and $\mathcal{H} = \int_M H^2 dM$ for surfaces in R^n have the same critical points and that $\mathcal{W}$ is conformally invariant [61]. As a consequence, the critical surfaces of the Willmore functional, i.e. *Willmore surfaces*, are also conformally invariant. Actually this was already known to Blaschke and

his students [4]. They called such surfaces "conformally minimal surfaces". Moreover, due to the work of Blaschke [4], Bryant [8], Ejiri [31], Willmore surfaces are related with conformally harmonic maps, via the *conformal Gauss map*. The seminal work of Bryant [8], [9], provided a modern treatment of Willmore surfaces by using moving frame methods and also algebraic geometry. Following Bryant's results Willmore surfaces received much attention and were investigated using various methods, see for example [5], [11], [15], [31], [34], [43], [46], [47], [52]. Although the geometric methods applied by Bryant have shown their power for the study of Willmore surfaces, many questions are still open, especially for Willmore surfaces in $S^n, n > 4$. Certainly one of the open problems is whether there exist new kinds of Willmore two spheres in S^n when $n > 4$.

The close relationship between Willmore surfaces and harmonic maps into symmetric spaces is an indication (also see [11]) that it may be useful to apply integrable system methods for the study of Willmore immersions. This is by far not a new idea and many authors have chosen this approach, see for example [2], [5], [34], [46], [64].

However, in [34] singularities will occur which are difficult to control and the other papers only deal with special questions. The authors of this paper are not aware of any framework studying Willmore surfaces in S^n by using loop group methods in a complete and satisfactory way. It is the goal of this paper to provide such a complete framework in which one is able to answer at least some of the still open questions for Willmore surfaces in S^n and to illustrate this work by examples, some of which are new.

The application of integrable system methods to geometric questions started in the 70's. In this paper we mainly use integrable system techniques in the context of harmonic maps from surfaces into symmetric spaces. The seminal paper by Uhlenbeck [57] leads to an understanding of harmonic 2-spheres in $U(n)$. The introduction of loop group methods and the realization of Bäcklund transformations as dressing action presented in [57] developed into a standard technique for the study of more general harmonic maps and their associated surface classes.

Uhlenbeck's treatment for harmonic 2-spheres was developed further by Burstall and Guest for all compact Lie groups and all compact inner symmetric spaces [12]. A somewhat different viewpoint was introduced in [26]. The use of extended frames was supplemented by the study of holomorphic extended frames which induce the same harmonic map. The Maurer-Cartan forms (also called "potentials") of such holomorphic frames is in some sense quite similar to the Weierstrass representation of minimal surfaces in R^3.

This method has been shown to be successful in the study of many surface classes. In these cases one characterizes surfaces classes by the harmonicity of certain "Gauss maps" and applies the loop group method to these Gauss maps, see for examples [6], [13], [20], [21], [23] and reference therein.

The main purpose of our paper is to build a framework for the global geometry of Willmore surfaces in S^{n+2} via the loop group method [26]. First it is well known that for any surface $y : M^2 \to S^{n+2}$, one can define globally a conformal map, called *conformal Gauss map*, $Gr : M^2 \to Gr_{1,3}R^{n+4}_1 = SO^+(1, n+3)/SO^+(1, 3) \times SO(n)$. A beautiful description of this topic can be found in [15] (see also [59] for the general theory for submanifolds). Then the theorem of Blaschke, Bryant, Ejiri, states that y is Willmore if and only if Gr is harmonic. So we have obtained a way to associate globally a conformally harmonic map to each Willmore surface into S^{n+2}. Actually, the potentials of conformal Gauss maps of Willmore surfaces satisfy a certain nilpotency condition ("strongly conformally harmonic maps").

How about the converse? As shown in Theorem 3.10 and Theorem 3.16, from a strongly conformally harmonic map $Gr : M^2 \to Gr_{1,3}R^{n+4}_1 = SO^+(1, n+3)/SO^+(1, 3) \times SO(n)$, we obtain, with the exception of some specific cases, a Willmore surface with Gr as its conformal Gauss map. This is the main topic in Section 3. Later, in section 5, we show that the strongly

conformally harmonic maps which are not associated with Willmore surfaces can be recognized by some property of their potential and can thus easily be avoided (see Theorem 5.2 for details).

Since we are particularly interested in Willmore spheres in S^{n+2} we characterize their potentials. In order to achieve this goal we show (see Theorem 4.21) that to every harmonic map $f : M \to G/K$, M any simply connected Riemann surface, G/K a non-compact inner symmetric space, one can construct a harmonic map $f_U : M \to U/(U \cap K^{\mathbb{C}})$, into the compact dual of G/K which has the same potential as the original harmonic map f. As a consequence of the work of Burstall and Guest [12] we obtain that every Willmore sphere in S^{n+2} is of finite uniton type. In particular, its potential only takes values in some nilpotent Lie algebra which can be described explicitly due to results of [12]. As a consequence of this we know exactly which potentials can yield Willmore spheres. The only issue remaining is whether there will be branch points (or even worse singularities). A detailed discussion of Willmore spheres in S^{n+1} is contained in [27]. We state an explicit example of a new, full, singularity free (isotropic) Willmore sphere in S^6 which is not S-Willmore.

As further examples we give Willmore surfaces with symmetries like equivariant Willmore surfaces and homogeneous Willmore surfaces. Some of these examples are new to the authors knowledge.

This paper is organized as follows: In Section 2 we recall the moving frame treatment of Willmore surfaces, following the methods of [15], relating a Willmore surface with its conformal Gauss map. We also briefly compare our treatment with Hélein's framework. Then we introduce the basic facts about harmonic maps and apply them to describe the conformal Gauss maps in Section 3. Moreover, also in this section, we describe a global correspondence between Willmore surfaces and conformally harmonic maps of a special type. In Section 4, we first recall the DPW method for harmonic maps into symmetric spaces. Some results, so far only proven for compact symmetric spaces, we generalize to our non-compact case. Section 5 is devoted to the application of our main results, including a description of the potentials corresponding to Willmore surfaces which are conformal to minimal surfaces in R^n. Moreover, by using Wu's formula (see section 4.3 and [63]), we show that isotropic Willmore surfaces in S^4 are Willmore surfaces of finite uniton type and, last not least, we present a concrete new non-S-Willmore Willmore two sphere in S^6. Finally, we discuss Willmore surfaces with symmetries. It turns out that many of the examples presentetd in this section admit a one-parameter group of symmetries. In the end of Section 5, we briefly discuss homogeneous Willmore surfaces (surfaces with a two-parameter group of symmetries) in our framework. At the end of the paper there are two appendices: the first one is to show that there are two open "big cells" of the Iwasawa decomposition for the twisted loop groups used in this paper; the other one is a short discussion concerning inner symmetric spaces.

2 Willmore surfaces in S^{n+2}

In [15], a natural and simple treatment of the conformal geometry of surfaces in S^{n+2} is presented. Particular emphasis is given to conformal immersions into S^3 and S^4 and to conformal immersions of tori into S^{n+2}. In this paper we will use the same set-up and give a description of a conformal surface in S^{n+2} by using the Maurer-Cartan form of some lift. We will review first the projective light cone model of the conformal geometry of S^{n+2} and derive the surface theory in this model. In view of our goal to describe Willmore surfaces, we then reformulate conformal surface theory on the Lie algebra level. After these preparations, we will briefly recall the basic and well-known description of Willmore surfaces.

2.1 Conformal surface theory in the projective light cone model

Let $\mathbb{R}^{n+4}_1$ denote Minkowski space, i.e. we consider $\mathbb{R}^{n+4}$ equipped with the Lorentzian metric

$$< x, y >= -x_0y_0 + \sum_{j=1}^{n+3} x_jy_j = x^t I_{1,n+3}y, \quad I_{1,n+3} = diag(-1, 1, \cdots, 1).$$

Let $\mathcal{C}^{n+3} = \{x \in \mathbb{R}^{n+4}_1 | < x, x >= 0, x_0 > 0\}$ denote the forward light cone of $\mathbb{R}^{n+4}_1$. It is easy to see that the projective light cone

$$Q^{n+2} = \{\ [x] \in \mathbb{R}P^{n+3} \mid x \in \mathcal{C}^{n+3} \setminus \{0\}\}$$

with the induced conformal metric, is conformally equivalent to S^{n+2}. Moreover, the conformal group of Q^{n+2} is exactly the projectivized orthogonal group $O(1, n + 3)/\{\pm 1\}$ of $\mathbb{R}^{n+4}_1$, acting on Q^{n+2} by

$$T([x]) = [Tx], \quad T \in O(1, n + 3).$$

By $SO^+(1, n + 3)$ we denote the connected component of the special linear isometry group of $\mathbb{R}^{n+4}_1$ which contains the identity element. Here " $+$" comes from the fact that $SO^+(1, n+3)$ preserves the forward timelike direction.

Then the Lie algebra of $O(1, n + 3)$ and $SO^+(1, n + 3)$ is

$$\mathfrak{so}(1, n + 3) = \mathfrak{g} = \{X \in gl(n + 4, \mathbb{R}) | X^t I_{1,n+3} + I_{1,n+3}X = 0\}.$$

Let $y : M \to S^{n+2}$ be a conformal immersion from a Riemann surface M. Let $U \subset M$ be a contractible open subset. A local lift of y is a map $Y : U \to \mathcal{C}^{n+3} \setminus \{0\}$ such that $\pi \circ Y = y$. Two different local lifts differ by a scaling, thus they induce the same conformal metric on M. Here we call y a *conformal* immersion, if $\langle Y_z, Y_z\rangle = 0$ and $\langle Y_z, Y_{\bar{z}}\rangle > 0$ for any local lift Y and any complex coordinate z on M. Noticing $\langle Y, Y_{z\bar{z}}\rangle = -\langle Y_z, Y_{\bar{z}}\rangle < 0$, we see that

$$V = \mathrm{Span}_{\mathbb{R}}\{Y, \mathrm{Re}Y_z, \mathrm{Im}Y_z, Y_{z\bar{z}}\} \tag{1}$$

is a rank-4 Lorentzian sub-bundle over U, and there is a natural decomposition $U \times \mathbb{R}^{n+4}_1 = V \oplus V^\perp$, where $V^\perp$ is the orthogonal complement of V. Note that both, V and $V^\perp$, are independent of the choice of Y and z, and therefore conformally invariant. In fact, we obtain a global conformally invariant bundle decomposition $M \times \mathbb{R}^{n+4}_1 = V \oplus V^\perp$. For any $p \in M$, we denote by V_p the fiber of V at p. And the complexifications of V and $V^\perp$ are denoted by $V_{\mathbb{C}}$ and $V^\perp_{\mathbb{C}}$ respectively.

Since Y takes values in the forward light cone $\mathcal{C}^{n+3}$, we are only interested in conformal transformations which are contained in $SO^+(1, n + 3)$.

Fixing a local coordinate z on U, there exists a local lift Y satisfying $|\mathrm{d}Y|^2 = |\mathrm{d}z|^2$, called the canonical lift (with respect to z). Given a canonical lift Y we choose the frame $\{Y, Y_z, Y_{\bar{z}}, N\}$ of $V_{\mathbb{C}}$, where N is the uniquely determined section of V over U satisfying

$$\langle N, Y_z\rangle = \langle N, Y_{\bar{z}}\rangle = \langle N, N\rangle = 0, \langle N, Y\rangle = -1. \tag{2}$$

Note that N lies in the forward light cone $\mathcal{C}^{n+3}$ and that $N \equiv 2Y_{z\bar{z}} \mod Y$ holds.

Next we define *the conformal Gauss map* of y.

Definition 2.1. *([8, 15, 31, 49]) Let $y : M \to S^{n+2}$ be a conformally immersed surface. The* conformal Gauss map *of y is defined as*

$$\begin{array}{rccl} Gr : & M & \to & Gr_{1,3}(\mathbb{R}^{n+4}_1) = SO^+(1, n + 3)/SO^+(1, 3) \times SO(n) \\ & p \in M & \mapsto & V_p \end{array} \tag{3}$$

Moreover, let Y be a canonical lift of y with respect to a local coordinate $z = u + iv$. Embedding $Gr_{1,3}(\mathbb{R}^{n+4}_1)$ into the exterior product $\Lambda^4\mathbb{R}^{n+4}_1$, we have

$$Gr = Y \wedge Y_u \wedge Y_v \wedge N = -2i \cdot Y \wedge Y_z \wedge Y_{\bar{z}} \wedge N$$

where N is the frame vector determined in (2)*.*

Given a (local) canonical lift Y we note that Y_{zz} is orthogonal to Y, Y_z and $Y_{\bar{z}}$. Therefore there exists a complex valued function s and a section $\kappa \in \Gamma(V^{\perp}_{\mathbb{C}})$ such that

$$Y_{zz} = -\frac{s}{2}Y + \kappa. \tag{4}$$

This defines two basic invariants of y: κ, called *the conformal Hopf differential* of y , and s, called *the Schwarzian* of y. Clearly, κ and s depend on the coordinate z (for a more detailed discussion, see [15, 49]).

Let D denote the $V^{\perp}_{\mathbb{C}}$ part of the natural connection of $\mathbb{C}^{n+4}$. Then for any section $\psi \in \Gamma(V^{\perp}_{\mathbb{C}})$ of the normal bundle and any (local) canonical lift Y of some conformal immersion y into S^n we obtain the structure equations ([15], [47]):

$$\begin{cases} Y_{zz} = -\frac{s}{2}Y + \kappa, \\ Y_{z\bar{z}} = -\langle\kappa,\bar{\kappa}\rangle Y + \frac{1}{2}N, \\ N_z = -2\langle\kappa,\bar{\kappa}\rangle Y_z - sY_{\bar{z}} + 2D_{\bar{z}}\kappa, \\ \psi_z = D_z\psi + 2\langle\psi, D_{\bar{z}}\kappa\rangle Y - 2\langle\psi,\kappa\rangle Y_{\bar{z}}, \end{cases} \tag{5}$$

For these structure equations the integrability conditions are the conformal Gauss, Codazzi and Ricci equations respectively ([15], [47]):

$$\begin{cases} \frac{1}{2}s_{\bar{z}} = 3\langle\kappa, D_z\bar{\kappa}\rangle + \langle D_z\kappa,\bar{\kappa}\rangle, \\ \mathrm{Im}(D_{\bar{z}}D_{\bar{z}}\kappa + \frac{\bar{s}}{2}\kappa) = 0, \\ R^D_{\bar{z}z} = D_{\bar{z}}D_z\psi - D_zD_{\bar{z}}\psi = 2\langle\psi,\kappa\rangle\bar{\kappa} - 2\langle\psi,\bar{\kappa}\rangle\kappa. \end{cases} \tag{6}$$

Choosing an orthonormal frame $\{\psi_j, j = 1, \cdots, n\}$ of the normal bundle $V^{\perp}$ over U, we can write the normal connection in the form

$$D_z\psi_j = \sum_{l=1}^{n} b_{jl}\psi_l, \quad b_{jl} + b_{lj} = 0.$$

Then, the conformal Hopf differential κ and its derivative $D_{\bar{z}}\kappa$ is of the form

$$\kappa = \sum_{j=1}^{n} k_j\psi_j, \ D_{\bar{z}}\kappa = \sum_{j=1}^{n} \beta_j\psi_j, \ \text{with } \beta_j = k_{j\bar{z}} - \sum_{j=1}^{n} \bar{b}_{jl}k_l, \ j = 1, \cdots, n.$$

Finally, setting

$$\phi_1 = \frac{1}{\sqrt{2}}(Y + N), \ \phi_2 = \frac{1}{\sqrt{2}}(-Y + N), \ \phi_3 = Y_z + Y_{\bar{z}}, \ \phi_4 = i(Y_z - Y_{\bar{z}}), \ k^2 = \sum_{j=1}^{n} |k_j|^2, \tag{7}$$

and defining the frame

$$F := (\phi_1, \phi_2, \phi_3, \phi_4, \psi_1, \cdots, \psi_n), \tag{8}$$

we obtain

Proposition 2.2. *Let $y : M \to S^{n+2}$ be a conformal immersion and Y its canonical lift over the open contractible set $U \subset M$. Then the frame F attains values in $SO^+(1, n+3)$, $F : U \to SO^+(1, n+3)$, and the Maurer-Cartan form $\alpha = F^{-1}dF$ of F is of the form*

$$\alpha = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} dz + \begin{pmatrix} \bar{A}_1 & \bar{B}_1 \\ \bar{B}_2 & \bar{A}_2 \end{pmatrix} d\bar{z},$$

with

$$A_1 = \begin{pmatrix} 0 & 0 & s_1 & s_2 \\ 0 & 0 & s_3 & s_4 \\ s_1 & -s_3 & 0 & 0 \\ s_2 & -s_4 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \vdots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{pmatrix}, \tag{9}$$

$$\begin{cases} s_1 = \frac{1}{2\sqrt{2}}(1 - s - 2k^2), \;\; s_2 = -\frac{i}{2\sqrt{2}}(1 + s - 2k^2), \\ s_3 = \frac{1}{2\sqrt{2}}(1 + s + 2k^2), \;\; s_4 = -\frac{i}{2\sqrt{2}}(1 - s + 2k^2), \end{cases} \tag{10}$$

$$B_1 = \begin{pmatrix} \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n \\ -k_1 & \cdots & -k_n \\ -ik_1 & \cdots & -ik_n \end{pmatrix}, \quad B_2 = \begin{pmatrix} \sqrt{2}\beta_1 & \sqrt{2}\beta_1 & k_1 & ik_1 \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{2}\beta_n & \sqrt{2}\beta_n & k_n & ik_n \end{pmatrix} = -B_1^t I_{1,3}. \tag{11}$$

Conversely, assume we have some frame $F = (\phi_1, \cdots, \phi_4, \psi_1, \cdots, \psi_{n+4}) : U \to SO^+(1, n+3)$ such that the Maurer-Cartan form $\alpha = F^{-1}dF$ of F is of the above form, then

$$y = \pi_0(F) =: [(\phi_1 - \phi_2)] \tag{12}$$

is a conformal immersion from U into $Q^{n+2} \cong S^{n+2}$ (with canonical lift $\frac{1}{\sqrt{2}}(\phi_1 - \phi_2)$).

Remark 2.3. Note that for any point in M the rank of B_1 is at most 2. The case $B_1 \equiv 0$ is equivalent to y being conformal to a round sphere. The case of (maximal) $rank(B_1) = 1$ will be discussed later.

2.2 Willmore surfaces and harmonicity

The conformal Hopf differential κ plays an important role in the investigation of Willmore surfaces. To show this, we first take a look at the transformation formula of κ. If w is another complex coordinate, then $Y_1 = Y \cdot |\frac{dw}{dz}|$ is a canonical lift with respect to w. So the corresponding Hopf differential κ_1 with respect to Y_1 is

$$\kappa_1 = \kappa \cdot \left(\frac{dz}{dw}\right)^2 / |\frac{dz}{dw}|. \tag{13}$$

A direct computation using (5) and (13) shows that the conformal Gauss map Gr induces a conformally invariant (possibly degenerate) metric

$$g := \frac{1}{4}\langle \mathrm{d}G, \mathrm{d}G \rangle = \langle \kappa, \bar{\kappa} \rangle |dz|^2$$

globally on M. Note that this metric degenerates at umbilical points of y, which are by definition the points where κ vanishes (For Willmore surfaces with umbilical lines, we refer to [2]).

Nevertheless, this metric can be used to define the Willmore functional.

Definition 2.4. *([15], [47]) The* Willmore functional *of y is defined as four times the area of M with respect to the metric above:*

$$W(y) := 2i \int_M \langle \kappa, \bar{\kappa} \rangle dz \wedge d\bar{z}. \tag{14}$$

An immersed surface $y : M \to S^{n+2}$ is called a Willmore surface*, if it is a critical point of the Willmore functional with respect to any variation (with compact support) of the map $y : M \to S^{n+2}$.*

From formula (13) it is straightforward to verify that $W(y)$ is well-defined. Moreover, suppose that $x : M \to \mathbb{R}^{n+2}$ is the stereographic projection of y into $\mathbb{R}^{n+2}$. Let $H,\ K$ denote the mean curvature and Gauss curvature of x. Then one can verify easily that

$$W(y) = W(x) = \int_M (H^2 - K) dM,$$

holds. Thus our definition of the Willmore functional coincides with the usual definition. It is well-known that Willmore surfaces can be characterized as follows [8, 15, 31, 59].

Theorem 2.5. *For a conformal immersion $y : M \to S^{n+2}$, the following three conditions are equivalent:*

(i) y is Willmore;

(ii) The conformal Gauss map Gr is a conformally harmonic map into $G_{3,1}(\mathbb{R}^{n+3}_1)$;

(iii) The conformal Hopf differential κ of y satisfies the "Willmore condition":

$$D_{\bar{z}} D_{\bar{z}} \kappa + \frac{\bar{s}}{2} \kappa = 0 \tag{15}$$

for any contractible chart of M.

Remark 2.6. Note that the Willmore condition is stronger than the conformal Codazzi equation (6).

Remark 2.7. The definitions and statements given above are correct, as long as the immersion y is sufficiently often differentiable. Considering the classical Willmore functional one observes that for the functional to make sense we only need that y is contained in the Sobolev space $W^{2,2}$. It has been shown by Kuwert and Schätzle (see [44] and references therein) that already under these very weak assumptions it follows that the immersion y is real analytic. As a consequence of this, the conformal Gauss map of any Willmore immersion is real analytic as well. We will therefore always assume w.l.g. that our immersions all are real analytic. Since we have shown just above that these Gauss maps are conformally harmonic, in this paper we will exclusively consider real analytic harmonic maps. Note also that a general result of Eells and Sampson [29] states that harmonic maps from surfaces with Riemannian metric are real analytic.

Now we introduce the notion of the so-called "dual Willmore surface", which is of essential importance in Bryant's and Ejiri's description of Willmore two-spheres.

Definition 2.8. *([8], [31], [49]) Let $y : M \to S^{n+2}$ be a Willmore surface with M_0 the set of umbilical points of y. A map $\hat{y} = [\hat{Y}] : M \setminus M_0 \to S^{n+2}$ is called a "dual Willmore surface" of y, if $\hat{Y}(p) \in V_p$ for any $p \in M \setminus M_0$ and either $\hat{y}$ is a point, or $\hat{y}$ has the same conformal Gauss map as y on $M \setminus M_0$, (and hence $\hat{y}$ is also Willmore on the points where it is an immersion).*

There exist many Willmore surfaces ([2], [7], [8], [31], [46], etc.) which admit dual Willmore surfaces. But in general a Willmore surface in S^{n+2} may not admit a dual surface. To describe Willmore surfaces having dual surfaces, Ejiri introduced the so-called *S-Willmore surfaces* in [31]. For this paper it is convenient to define S-Willmore surfaces as follows (see also [48]):

Definition 2.9. *([31]) A Willmore immersion $y : M \to S^n$ is called an S-Willmore surface if on any open subset U, away from the umbilical points, the conformal Hopf differential κ of y satisfies*

$$D_{\bar{z}}\kappa || \kappa,$$

i.e. there exists some function μ on U such that $D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa = 0$.

Corollary 2.10. *Let y be a Willmore surface which is not totally umbilical. Then y is S-Willmore if and only if the (maximal) rank of B_1 in Proposition 2.2 is 1.*

Theorem 2.11. ([8], [31], [48], [60]) *A (non totally umbilical) Willmore surface y is S-Willmore if and only if it has a unique dual (Willmore) surface except at the umbilical points.*

To deal with umbilical points, we need a technical lemma on complex Ricatti equations, which appears naturally in the study of Willmore surfaces (see e.g. (4.8) in [47] and (64) in [34]), as well as in other fields. The Ricatti equation can be written as

$$\mu_z - \frac{\mu^2}{2} - s = 0. \tag{16}$$

In general, μ can be written in the form $\mu = -\frac{2\nu_z}{\nu}$. And the equation (16) is equivalent with the linear equation

$$\nu_{zz} + \frac{s}{2}\nu = 0. \tag{17}$$

Lemma 2.12. *Let $\mathbb{D} \subset \mathbb{C}$ be a contractible open subset of $\mathbb{C}$ containing 0. Let $\nu = \nu(z, w)$ and $s = s(z, w)$ be two holomorphic functions for any $(z, w) \in \mathbb{D} \times \mathbb{D}$ satisfying (17), with $\nu(0, 0) = 0$ and $\nu \not\equiv 0$. Then the limit of the function $\mu = -\frac{2\nu_z}{\nu}$ exists when $(z, w) \to (0, 0)$, i.e., it is a finite number or ∞.*

Proof. Let ν_1 and ν_2 be two solutions of (17) satisfying the initial conditions

$$\begin{cases} \nu_1(0, w) = 1, \\ \nu_{1z}(0, w) = 0, \end{cases} \qquad \begin{cases} \nu_2(0, w) = 0, \\ \nu_{2z}(0, w) = 1, \end{cases}$$

respectively for every $w \in \mathbb{D}$. Then every solution ν of (17) is of the form $\nu = \tau_1(w) \cdot \nu_1(z, w) + \tau_2(w) \cdot \nu_2(z, w)$, with $\tau_1(w)$, $\tau_2(w)$ holomorphic functions in w. The condition $\nu(0, 0) = 0$ yields $\tau_1(0) = 0$. If $\nu_z(0, 0) \neq 0$, then μ goes to infinity when $(z, w) \to (0, 0)$. So we only need to consider the case $\nu_z(0, 0) = 0$. Hence we obtain $\tau_2(0) = 0$. Now write $\tau_1(w)$ and $\tau_2(w)$ in the form $\tau_1 = w^{m_1}h_1(w)$ and $\tau_2 = w^{m_2}h_2(w)$ with $m_1, m_2 \in \mathbb{Z}^+$, $h_1(w)$, $h_2(w)$ holomorphic functions and $h_1(0)h_2(0) \neq 0$. Hence μ is of the form

$$\mu = -2\frac{w^{m_1}h_1(w)\nu_{1z}(z, w) + w^{m_2}h_2(w)\nu_{2z}(z, w)}{w^{m_1}h_1(w)\nu_1(z, w) + w^{m_2}h_2(w)\nu_2(z, w)}.$$

It is easy to verify now that in the limit $(z, w) \to (0, 0)$, the function μ tends to $-2\frac{h_2(0)}{h_1(0)} \neq 0$ if $m_1 = m_2$, it tends to ∞ if $m_1 < m_2$, and it tends to 0 if $m_1 > m_2$. $\square$

After these preparations we will give a proof of the global existence of dual Willmore surfaces for S-Willmore surfaces. Note that in the proofs on the duality theorems of Willmore surfaces given so far in the literature, it stays unclear what will happen at umbilical points.

Theorem 2.13. *Let f be the conformal Gauss map of a non totally umbilical Willmore surface $y : M \to S^{n+2}$. Assume that f is also the conformal Gauss map of a Willmore surface $\hat{y}$ on an open and dense subset of M.*

(a) If y is S-Willmore, then $\hat{y}$ is congruent to y or dual to y. When $\hat{y}$ is the dual surface of y, $\hat{y}$ is well-defined at the umbilical points of y.

(b) If y is not S-Willmore, then $\hat{y}$ is congruent to y.

Proof. Since y is not totally umbilical, we assume that y has a non-zero conformal Hopf differential on an open dense subset M_1 of M. If $\hat{y}$ is different from y on M_1, for any open contractible subset U with canonical lift (Y, z) of y, we can assume that on $U \cap M_1$ a lift of $\hat{y}$ is of the form (see also (4.1) in [47])

$$Y_\mu = N + \bar{\mu} Y_z + \mu Y_{\bar{z}} + \frac{|\mu|^2}{2} Y,$$

i.e., $\hat{y} = [Y_\mu]$. By using (5), a straightforward computation shows that

$$Y_{\mu z} = 2D_{\bar{z}}\kappa + \bar{\mu}\kappa \mod \{Y_0, N, Y_z, Y_{\bar{z}}\}$$

holds (See also (4.2), (4.3) in [47]).

(a) If y is S-Willmore, there exists a unique μ such that $2D_{\bar{z}}\kappa + \bar{\mu}\kappa = 0$ on $U \cap M_1$. And it is easy to prove that $[Y_\mu]$ is Willmore and has f as its conformal Gauss map. For a definition of $[Y_\mu]$ at the umbilical points $U \backslash (U \cap M_1)$, we first observe that substituting $2D_{\bar{z}}\kappa + \bar{\mu}\kappa = 0$ into the Willmore condition (15) on $U \cap M_1$ yields that μ is a solution to the Ricatti equation

$$\mu_z - \frac{\mu^2}{2} - s = 0.$$

Note now that $s = s(z, \bar{z})$ is a real analytic function. Therefore $s = s(z, w)$ is holomorphic in z and w for z and w sufficiently small. For any point $p \in U \backslash (U \cap M_1)$, we may assume that $z(p) = 0$ by changing coordinates if necessary. Therefore, by Lemma 2.12, we have that at the point p the function μ always has a limit, a finite number or infinity. If μ is a finite number, Y_μ is well-defined as before. If μ goes to infinity,

$$\hat{y} = [Y_\mu] = \lim_{\mu \to \infty} \left[\frac{2}{|\mu|^2} N + \frac{2}{\mu} Y_z + \frac{2}{\bar{\mu}} Y_{\bar{z}} + Y \right] = [Y] = y.$$

This implies that $[Y_\mu]$ is also well-defined as above on the umbilical points $M \backslash M_1$ and therefore globally defined on M.

(b) If y is a non S-Willmore Willmore surface, there exists no μ such that $D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa = 0$ by definition. Hence for any μ, Y_μ can not share the same conformal Gauss map with y. Noticing that up to a scaling, any lift of $\hat{y}$ is either of the form Y_μ or equal to Y, the uniqueness of y follows. $\square$

Remark 2.14. Note that in terms of Proposition 2.2 the condition in (a) is equivalent with $rank B_1 = 1$ and the condition in (b) is equivalent with $rank B_1 = 2$.

We will say "the conformal Gauss map contains a constant lightlike vector Y_0" if there exists a non-zero constant lightlike vector Y_0 in $\mathbb{R}^{n+4}_1$ satisfying $Y_0 \in V_p$ for all $p \in M$. Then a well-known fact states (one can find a proof in [50])

Theorem 2.15. *A Willmore surface y is conformal to a minimal surface in R^{n+2} if and only if its conformal Gauss map Gr contains a constant lightlike vector.*

There exist Willmore surfaces which fail to be immersions at some points. To include surfaces of this type, we introduce the notion of *Willmore maps* and *strong Willmore maps*.

Definition 2.16. *A smooth map y from a Riemann surface M to S^{n+2} is called a Willmore map if it is a conformal Willmore immersion on an open dense subset $\hat{M}$ of M. The points in $M_0 = M \backslash \hat{M}$ are called branch points of y, at which points y fails to be an immersion.*

Moreover, y is called a strong Willmore map if the conformal Gauss map $Gr : \hat{M} \rightarrow SO^+(1, n+3)/SO^+(1,3) \times SO(n)$ of y can be extended smoothly (and hence real analytically) to M.

Remark 2.17. It is an interesting (open) problem under which conditions a Willmore map will be a strong Willmore map.

Example 2.18. 1. Let $x : \hat{M} \rightarrow \mathbb{R}^n$ be a complete minimal surface with finite total curvature. By the classical theory of minimal surfaces, x is an algebraic minimal surface with finitely many ends $\{p_1, \cdots, p_r\}$. And by the inverse of the stereographic projection x becomes a smooth map y from a compact Riemann surface $M = \hat{M} \cup \{p_1, \cdots, p_r\}$ to S^n. If all the ends of x are embedded planar ends, y will be a Willmore immersion. If some ends $\{p_{j_1}, \cdots, p_{j_t}\}$ fail to be embedded planar ends [8], [9], y will be a strong Willmore map with branch points $\{p_{j_1}, \cdots, p_{j_t}\}$.

2. Another interesting type of Willmore surfaces consists of the so-called isotropic (or super-conformal [7]) surfaces in S^4, which can be lifted to holomorphic or anti-holomorphic curves in the twistor bundle of S^4. It is well known now (see [31, 52, 51] for example) that such surfaces of genus 0, together with minimal surfaces in $\mathbb{R}^4$ with embedded planar ends of genus 0, provide all the possibilities of Willmore two spheres in S^4.

3. It is well-known that minimal surfaces in Riemannian space forms are Willmore surfaces ([8, 43, 59]). These surfaces are basic examples of Willmore surfaces. Moreover, they are S-Willmore surfaces, see [31, 49]. It will therefore be particularly of interest and importance to construct non-S-Willmore surfaces.

4.The first non-minimal Willmore surface was given by Ejiri in [30]. This non-S-Willmore Willmore surface actually was a torus in S^5. Later, using the Hopf bundle, Pinkall produced a family of non-minimal Willmore tori in S^3 via elastic curves ([54]).

Remark 2.19. (*Hélein and Ma's harmonic maps*)

In [34], Hélein followed the treatment of Bryant [8] to deal with Willmore surfaces in S^3 by using loop group methods [26]. He used two kinds of harmonic maps: the conformal Gauss map and the ones he first introduced as "roughly harmonic maps". In terms of the notation used here, for a Willmore immersion y in S^3 with local lift Y, let $\hat{Y} \in \Gamma(V)$ such that $\langle \hat{Y}, \hat{Y} \rangle = 0$, and $\langle Y, \hat{Y} \rangle = -1$. Then Hélein's roughly harmonic map is defined by

$$\mathfrak{H} = Y \wedge \hat{Y} : M \rightarrow Gr_{1,1}(R^5_1). \tag{18}$$

The reason of the name "roughly harmonic" is that although $\mathfrak{H}$ may not be harmonic in general, it really provides another family of flat connections (see (36) page 350 in [34] for details). If one assumes furthermore that $\hat{Y}$ satisfies

$$\langle \hat{Y}_z, Y_z \rangle = 0, \tag{19}$$

$\mathfrak{H}$ will be a harmonic map. Especially, for the Willmore surfaces y in S^3, there always exists a dual surface (Recall Definition 2.8 or see [8]). When $\hat{Y}$ is chosen as the lift of the dual surface $\hat{y}$ of y, one obtains an interesting harmonic map connecting the original surface and its dual surface. It is straightforward to generalize Hélein's notion of roughly harmonic maps to the case of Willmore immersions into S^{n+2}, since the definition above does not involve the co-dimensional information. Such natural generalizations following Hélein have been worked out in [64], by using the treatment of [59] on Willmore submanifolds.

In a different development, in [47], Ma considered the generalization of the notion of a dual surface for a Willmore surface y in S^{n+2}. Let $\hat{Y} \in \Gamma(V)$ be such that $\langle \hat{Y}, \hat{Y} \rangle = 0$, and

$\langle Y, \hat{Y}\rangle = -1$. Ma found that if $\hat{Y}$ satisfies

$$\begin{cases} \langle \hat{Y}_z, Y_z\rangle = 0, \\ \langle \hat{Y}_z, \hat{Y}_z\rangle = 0, \end{cases} \tag{20}$$

then $[\hat{Y}]$ is a new Willmore surface (may degenerate to a point, see [47]). In this case $[\hat{Y}]$ is called "an adjoint surface" of y. Different from dual surfaces, the adjoint surface $[\hat{Y}]$ is in general not unique (a detailed discussion on this can be found in [47]). Moreover, Ma showed that for an adjoint surface $[\hat{Y}]$ the map $\mathfrak{H} = Y \wedge \hat{Y} : M \to Gr_{1,1}(R^{n+3}_1)$ is a conformal harmonic map. One therefore obtains that this harmonic map defined by Ma is just a special case of Hélein's harmonic maps in [34], [64]. Ma's harmonic map may be a particularly natural generalization.

Note that for Hélein's harmonic map as well as for Ma's adjoint surfaces, it is usually not possible to prove global existence, since the solution of the equation (19) may have singularities. So it does not seem to be easy to use this approach to discuss the global problem directly. To be more concrete, first we would like to point out that (19) is exactly the Ricatti equation (16)

$$\mu_z - \frac{\mu^2}{2} - s = 0.$$

By Lemma 2.12, μ may take the value ∞ at some points. Therefore as in the proof of Theorem 2.13, at the points where μ approaches ∞, we have $[\hat{Y}] = [Y]$. This implies that the 2-dimensional Lorenzian bundle $Span_{\mathbb{R}}\{Y,\ \hat{Y}\}$ defined by Y and $\hat{Y}$ reduces to a 1-dimensional lightlike bundle at these points. It stays unknown how to deal with the global properties for this kind of harmonic maps by using Hélein's approach. This is one of the reasons why we use the conformal Gauss map to study Willmore surfaces, although the computations using Hélein's harmonic map would perhaps be somewhat easier.

The relation between our approach and Hélein's is very interesting, in particular in view of Ma's contributions. We hope to be able to pursue this in a subsequent publication.

3 Conformally harmonic maps into $SO^+(1, n+3)/SO^+(1, 3)\times SO(n)$

In this section, we first review the basic description of harmonic maps. Then we apply it to the harmonic maps into $SO^+(1, n+3)/SO^+(1, 3) \times SO(n)$. We have seen above that Willmore surfaces are related to conformally harmonic maps with special Maurer-Cartan forms. Since not every conformally harmonic map is the conformal Gauss map of some strong Willmore map, we give a necessary and sufficient condition for a conformally harmonic map to be the conformal Gauss map of a strong Willmore map.

3.1 Harmonic maps into Symmetric space G/K

Let $N = G/K$ be a symmetric space with involution $\sigma : G \to G$ such that $G^\sigma \supset K \supset (G^\sigma)_0$. Let $\mathfrak{g}$ and $\mathfrak{k}$ demote the Lie algebras of G and K respectively. The involution σ induces the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Let $\pi : G \to G/K$ denote the projection of G onto G/K.

Let $f : M \to G/K$ be a conformally harmonic map from a connected Riemann surface M. Let $U \subset M$ be an open contractible subset. Then there exists a frame $F : U \to G$ such that $f = \pi \circ F$ on U. Let α denote the Maurer-Cartan form of F. Then α satisfies the Maurer-Cartan equation and altogether we have

$$F^{-1}dF = \alpha, \quad \text{and} \quad d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Decomposing α with respect to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ we obtain

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}}, \quad \text{and} \quad \alpha_{\mathfrak{k}} \in \Gamma(\mathfrak{k} \otimes T^*M), \quad \text{and} \quad \alpha_{\mathfrak{p}} \in \Gamma(\mathfrak{p} \otimes T^*M),$$

Moreover,

$$\begin{cases} d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + \frac{1}{2}[\alpha_{\mathfrak{p}} \wedge \alpha_{\mathfrak{p}}] = 0, \\ d\alpha_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{p}}] = 0, \end{cases}$$

holds. Next we decompose $\alpha_{\mathfrak{p}}$ further into the $(1,0)-$part $\alpha'_{\mathfrak{p}}$ and the $(0,1)-$part $\alpha''_{\mathfrak{p}}$, and set

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{p}}, \qquad \lambda \in S^1. \tag{21}$$

With this notation we have

Lemma 3.1. *([26]) The map $f : M \to G/K$ is harmonic if and only if*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \quad \textit{for all } \lambda \in S^1. \tag{22}$$

Definition 3.2. *Let $f : M \to G/K$ be harmonic and α_λ the differential one-form defined above. Since, by the lemma, α_λ satisfies the integrability condition* (22)*, we consider on any contractible open subset $U \subset M$ the solution $F(z,\lambda)$ to the equation*

$$dF(z,\lambda) = F(z,\lambda)\alpha_\lambda$$

with the initial condition $F(z_0,\lambda) = e$, where z_0 is a fixed base point $z_0 \in U$, and e is the identity element in G. The map $F(z,\lambda)$ is called the extended frame *of the harmonic map f normalized at the base point $z = z_0$. Note that F satisfies $F(z,\lambda = 1) = F(z)$.*

Consider the complexification $TM^{\mathbb{C}} = T'M \oplus T''M$ and write $d = \partial + \bar{\partial}$. Then the lemma above can be restated

Lemma 3.3. *([26]) The map $f : M \to G/K$ is harmonic if and only if*

$$\begin{cases} d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + \frac{1}{2}[\alpha_{\mathfrak{p}} \wedge \alpha_{\mathfrak{p}}] = 0, \\ \bar{\partial}\alpha'_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}] = 0. \end{cases} \tag{23}$$

3.2 Harmonic maps into $SO^+(1,n+3)/SO^+(1,3) \times SO(n)$

Let's consider again $\mathbb{R}^{n+4}_1$, with the metric introduced in Section 2. Recall that by $SO^+(1,n+3)$ we denote the connected component of the special linear isometry group of $\mathbb{R}^{n+4}_1$ which contains the identity element. Here " $^+$" comes from the fact that $SO^+(1,n+3)$ preserves the forward timelike direction. Moreover, by $\mathfrak{so}(1,n+3) = \mathfrak{g} = \{X \in gl(n+4,\mathbb{R}) | X^t I_{1,n+3} + I_{1,n+3}X = 0\}$ we denote the Lie algebra of $SO^+(1,n+3)$.

Consider the involution

$$\begin{array}{rl} \sigma : SO^+(1,n+3) & \to SO^+(1,n+3) \\ A & \mapsto D^{-1}AD, \end{array} \tag{24}$$

with

$$D = \begin{pmatrix} -I_4 & 0 \\ 0 & I_n \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix. Then the fixed point group $SO^+(1,n+3)^\sigma$ of σ contains $SO^+(1,3) \times SO(n)$, where $SO^+(1,3)$ denotes a connected group according to our

convention. Moreover we have $SO^+(1,n+3)^\sigma \supset SO^+(1,3)\times SO(n) = (SO^+(1,n+3)^\sigma)_0$, where the subscript 0 denotes the connected component containing the identity element.

On the Lie algebra level we obtain

$$\mathfrak{g} = \left\{ \left(\begin{array}{cc} A_1 & B_1 \\ -B_1^t I_{1,3} & A_2 \end{array} \right) \mid A_1^t I_{1,3} + I_{1,3} A_1 = 0, \quad A_2 + A_2^t = 0 \right\},$$

$$\mathfrak{k} = \left\{ \left(\begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right) \mid A_1^t I_{1,3} + I_{1,3} A_1, \quad A_2 + A_2^t = 0 \right\},$$

$$\mathfrak{p} = \left\{ \left(\begin{array}{cc} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{array} \right) \right\}.$$

Now let $f : M \to SO^+(1,n+3)/SO^+(1,3)\times SO(n)$ be a harmonic map with local frame $F : U \to SO^+(1,n+3)$ and Maurer-Cartan form α on some contractible open subset U of M. Let z be a local complex coordinate on U. Writing

$$\alpha'_{\mathfrak{k}} = \left(\begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right) dz, \quad \text{and} \quad \alpha'_{\mathfrak{p}} = \left(\begin{array}{cc} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{array} \right) dz,$$

the harmonic map equations can be rephrased equivalently in the form

$$\begin{cases} Im\left(A_{1\bar z} + \bar A_1 A_1 - \bar B_1 B_1^t I_{1,3}\right) = 0, \\ Im\left(A_{2\bar z} + \bar A_2 A_2 - \bar B_1^t I_{1,3} B_1\right) = 0, \\ B_{1\bar z} + \bar A_1 B_1 - B_1 \bar A_2 = 0. \end{cases}$$

In section 2 we have seen that the Maurer Cartan form of the frame associated with a Willmore surface in S^{n+2} has a very special form. It is very fortunate that it is easy to detect, when such a special form can be obtained by gauging. We will see below that a crucial part of our paper is to bring B_1 into a canonical form.

Lemma 3.4. *For a $4\times n$ complex matrix B_1, there exists some $A \in SO^+(1,3)$ and there exist functions $\beta_j,\ k_j,\ j = 1,\cdots,n$ such that*

$$AB_1 = \left(\begin{array}{ccc} \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n \\ -k_1 & \cdots & -k_n \\ \pm i k_1 & \cdots & \pm i k_n \end{array} \right)$$

if and only if

$$B_1^t I_{1,3} B_1 = 0. \tag{25}$$

If B_1 is a real analytic matrix function defined on some contractible Riemann surface U, then A can be chosen globally on U as a real analytic matrix function such that on an open dense subset $\hat U$ of U one of the following two canonical forms is obtained:

$$AB_1 = \left(\begin{array}{ccc} \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n \\ -k_1 & \cdots & -k_n \\ -i k_1 & \cdots & -i k_n \end{array} \right) \ \text{on } \hat U, \ \text{or } AB_1 = \left(\begin{array}{ccc} \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n \\ -k_1 & \cdots & -k_n \\ i k_1 & \cdots & i k_n \end{array} \right) \ \text{on } \hat U.$$

Remark 3.5. Recall that from Proposition 2.2 that $k_j \equiv 0$ for all j on an open subset implies that the initial surface is umbilical, which is not of interest for Willmore surfaces.

Proof. Suppose that AB_1 is of the form stated in the lemma. Then

$$0 = (AB_1)^t I_{1,3} AB_1 = B_1^t A^t I_{1,3} AB_1 = B_1^t I_{1,3} B_1.$$

Conversely, assume $B_1^t I_{1,3} B_1 = 0$. Considering $B_1^t I_{1,3} B_1$ as the matrix product $(B_1^t I_{1,3}) \cdot B_1$ we obtain by Sylvester's rank inequality and the assumption (25)

$$rank(B_1^t I_{1,3}) + rank(B_1) \leq 4.$$

We obviously also have $rank(B_1^t I_{1,3}) = rank(B_1)$. Therefore $rank(B_1) \leq 2$. From $B_1^t I_{1,3} B_1 = 0$ and $rank(B_1) \leq 2$ we infer that the column vectors of B_1 are spanned by (at most) two non-zero orthogonal light-like vectors in $\mathbb{R}^4_1 \otimes \mathbb{C}$ given the natural complex linear inner product, i.e.,

$$B_1 = \begin{pmatrix} l_1 & l_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where

$$v_1^t,\ v_2^t \in \mathbb{C}^n, \quad l_1, l_2 \in \mathbb{R}^4_1 \otimes \mathbb{C}, \quad \text{and} \quad < l_i, l_j >= 0 \ \text{ for } \ i, j = 1, 2.$$

It is easy to see that a maximal subspace of $\mathbb{R}^4_1 \otimes \mathbb{C}$ containing only null vectors is 2-dimensional.

It is also an elementary computation to verify that every (complex) null vector can be mapped by some element of the real group $SO^+(1,3)$ into a (possibly complex) multiple of the vector $(1,-1,0,0)^t$ or the vector $(0,0,-1,\pm i)^t$.

As a consequence, if the rank of B_1 is 1, then we have proven the claim (even with much more special vectors). If the rank of B_1 is 2, then we choose two linearly independent column vectors l_1 and l_2. We can assume, by what was said above, that l_1 is one of the two special vectors stated above. Now it is very straightforward to verify that $\langle l_i, l_j \rangle = 0, i, j = 1, 2$ implies that l_2 is of the required from. Actually, by replacing l_2 by a vector of the form $l_1 + cl_2$, c some complex number, one can assume that l_1 and l_2 are multiples of the two special vectors above.

Note that we only need to prove the last statement when $rank B_1 = 2$ on an open subset of U, since the other cases are similar. The real analyticity of B_1 on U yields that $rank B_1 = 2$ on an open dense subset $\hat{U}$ of U. Without loss of generality we assume that $l_1, l_2 \neq 0$, $\langle l_1, \bar{l}_1 \rangle \equiv 0$, $\langle l_2, \bar{l}_2 \rangle > 0$ on $\hat{U}$. Then there exists a real analytic matrix function $\tilde{A} : \hat{U} \to O^+(1,3)$ such that

$$\tilde{A} l_1 \text{ takes values in } \operatorname{span}_{\mathbb{C}}\{(1,-1,0,0)^t\},\ \tilde{A} l_2 \text{ takes values in } \operatorname{span}_{\mathbb{C}}\{(0,0,1,i)^t\}.$$

Although $\hat{U}$ may not be connected now, we claim that the determinant $\det A$ of A takes the same value on all of $\hat{U}$, that is, $\det \tilde{A} = 1$ on $\hat{U}$ or $\det \tilde{A} = -1$ on $\hat{U}$. Then our lemma follows by setting $A = \tilde{A}$ when $\det \tilde{A} = 1$ and $A = \tilde{A} \cdot \operatorname{diag}(1,1,1-1)$ when $\det \tilde{A} = -1$.

To prove the claim, we consider any connected compact subset $\hat{U}^*$ of U which has a non-empty interior. We only need to show $\det \tilde{A}$ takes the same value at all of $\hat{U}^*$. Let $p, q \in \hat{U}^*$ be any two points. Then at an open subset $U_p \subset \hat{U}^*$ containing p,

$$\tilde{A} l_1 \text{ takes values in } \operatorname{span}_{\mathbb{C}}\{(1,-1,0,0)^t\}, \text{ and } \tilde{A} l_2 \text{ takes values in } \operatorname{span}_{\mathbb{C}}\{(0,0,1,i)^t\}.$$

Let $\gamma_{pq}(t) \subset U$ be a real analytic curve connecting p and q. Restricted to γ_{pq}, the vector functions $l_1, l_2 : U \to \mathbb{C}^4$ become real analytic vector functions depending on t. Therefore, there exist only finitely many points where l_1 or l_2 vanishes. For $j = 1, 2$, if at some point t_0 we have $l_j = 0$, then we will have that $l_j = (t - t_0)^m \hat{l}_j$ with $\hat{l}_j(t_0) \neq 0$ since l_j is real analytic. Hence we may choose some $\hat{l}_1, \hat{l}_2$ such that they are non-zero on γ_{pq}. Hence we can find a real analytic matrix function $\hat{A} : \gamma_{pq} \subset \hat{U}^* \to SO^+(1,3)$ such that $\hat{A} = \tilde{A}$ on $U_p \cap \gamma_{pq}$, and

$$\hat{A}\hat{l}_1 \ (\text{ and hence } \hat{A} l_1) \ \text{ takes value in } \operatorname{span}_{\mathbb{C}}\{(1,-1,0,0)^t\},$$

and

$$\hat{A}\hat{l}_2 \text{ (and hence } \hat{A}l_2) \text{ takes value in } \mathrm{span}_{\mathbb{C}}\{(0,0,1,i)^t\}.$$

The claim now follows from the equalities

$$\det \tilde{A}_p = \det \hat{A}_p,\ \det \hat{A}_p = \det \hat{A}_q,\ \det \tilde{A}_q = \det \hat{A}_q.$$

which follow from the fact that $\tilde{A}, \hat{A}$ induce the same orientation on $\mathbb{R}^4_1$. □

Lemma 3.6. *Let U be a contractible open Riemann surface. Let $f : U \to SO^+(1,n+3)/SO^+(1,3)\times SO(n)$ be a non-constant harmonic map with two frames $F,\ \hat{F} : U \to SO^+(1,n+3)$ and Maurer-Cartan forms $\alpha, \hat{\alpha}$. Using a local complex coordinate z on U, we write*

$$\alpha'_{\mathfrak{p}} = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{pmatrix} dz, \qquad \hat{\alpha}'_{\mathfrak{p}} = \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz.$$

Then

$$B_1^t I_{1,3} B_1 = 0 \quad \text{if and only if} \quad \hat{B}_1^t I_{1,3} \hat{B}_1 = 0.$$

Moreover, under the above condition, we have that

$$\bar{B}_1^t I_{1,3} B_1 = 0 \quad \text{if and only if} \quad \bar{\hat{B}}_1^t I_{1,3} \hat{B}_1 = 0.$$

Proof. Since F and $\hat{F}$ are lifts of the same harmonic map f, there exists

$$F_0 = \begin{pmatrix} F_{01} & 0 \\ 0 & F_{02} \end{pmatrix} : U \to SO^+(1,3)\times SO(n)$$

such that $\hat{F} = F \cdot F_0$. It turns out that $\hat{\alpha} = F_0^{-1}\alpha F_0 + F_0^{-1} dF_0$, yielding $\hat{B}_1 = F_{01}^{-1} B_1 F_{02}$. So

$$\hat{B}_1^t I_{1,3} \hat{B}_1 = F_{02}^{-1} B_1^t F_{01}^{-1,t} I_{1,3} F_{01}^{-1} B_1 F_{02} = F_{02}^{-1} B_1^t I_{1,3} B_1 F_{02}.$$

The last statement comes from the fact that F_{01} and F_{02} are real matrices. □

Definition 3.7. *Let $f : M \to SO^+(1,n+3)/SO^+(1,3)\times SO(n)$ be a harmonic map. We call f a* **strongly conformally harmonic map** *if for any point $p \in M$, there exists a neighborhood U_p of p and a frame F (with Maurer-Cartan form α) of y on U_p satisfying*

$$B_1^t I_{1,3} B_1 = 0, \text{ where } \alpha'_{\mathfrak{p}} = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{pmatrix} dz. \tag{26}$$

Applying this definition to Willmore surfaces, we derive immediately

Corollary 3.8. *The conformal Gauss map of a strong Willmore map is a strongly conformally harmonic map.*

Lemma 3.9. *Let U be a contractible open Riemann surface with local complex coordinate z. Let $f : U \to SO^+(1,n+3)/SO^+(1,3)\times SO(n)$ be a strongly conformally harmonic map with frame $F : U \to SO^+(1,n+3)$ and Maurer-Cartan form α. Set*

$$\alpha'_{\mathfrak{k}} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} dz, \qquad \alpha'_{\mathfrak{p}} = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{pmatrix} dz.$$

Then B_1 has, after some gauge or a change of orientation if necessary, the form

$$B_1 = \begin{pmatrix} \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n \\ -k_1 & \cdots & -k_n \\ -ik_1 & \cdots & -ik_n \end{pmatrix}, \tag{27}$$

on an open dense subset $\hat{U}$ of U.

Proof. The conformal harmonicity of f ensures that B_1 is a real analytic matrix function [29], [44]. By Lemma 3.4, there exists $A:\hat{U}\rightarrow SO^+(1,3)$ such that

$$AB_1=\begin{pmatrix}\sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n\\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n\\ -k_1 & \cdots & -k_n\\ -ik_1 & \cdots & -ik_n\end{pmatrix}\ \text{ on } \hat{U},\ \text{ or } AB_1=\begin{pmatrix}\sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n\\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n\\ -k_1 & \cdots & -k_n\\ ik_1 & \cdots & ik_n\end{pmatrix}\ \text{ on } \hat{U}.$$

For the first case, setting $\hat{F}=F\cdot\mathrm{diag}(A,I_n)$ we obtain $\hat{B}_1=AB_1$ on $\hat{U}$.

For the second case, setting $w=\bar{z}$ induces an opposite orientation on U and U is also a Riemann surface for this new coordinate. Now $A\bar{B}_1$ is of the desired form. □

Before introducing the main results of this section, we first consider a class of maps with specific geometric meaning. Let f be a map from M into $SO^+(1,n+3)/SO^+(1,3)\times SO(n)=Gr_{1,3}(\mathbb{R}^{n+4}_1)$. Assume that $f(p)=V_p$, $p\in M$, with $V_p\subset\mathbb{R}^{n+4}_1$. We will say that f *contains a constant light-like vector* Y_0 if there exists a non-zero constant lightlike vector Y_0 in $\mathbb{R}^{n+4}_1$ satisfying $Y_0\in V_p$ for all $p\in M$. Note that from the viewpoint of Möbius geometry ([15], [40]), f is also a 2-sphere congruence in S^{n+2}. Under some condition ([48], [49]), f will envelope one (or a pair of) conformal surface(s) in S^{n+2}. In this case f contains a constant light-like vector Y_0 if and only if an enveloping surface reduces to the point $[Y_0]$.

Theorem 3.10. *We retain the assumptions and notation of Lemma 3.9. Then B_1 has the form* (27) *and f is a conformally harmonic map on U. Writing A_1 in the form*

$$A_1=\begin{pmatrix}0 & a_{12} & a_{13} & a_{14}\\ a_{12} & 0 & a_{23} & a_{24}\\ a_{13} & -a_{23} & 0 & a_{34}\\ a_{14} & -a_{24} & -a_{34} & 0\end{pmatrix},$$

we distinguish two cases:

(a) $a_{13}+a_{23}\not\equiv 0$ on U: In this case, there exists an open dense subset $U\backslash U_0$ such that $a_{13}+a_{23}\neq 0$ on $U\backslash U_0$ and $a_{13}+a_{23}=0$ on U_0. Then on $U\backslash U_0$ the map f is the conformal Gauss map of a Willmore surface $y:U\backslash U_0\rightarrow S^{n+2}$ and y is not an immersion on U_0. Moreover, y is S-Willmore if and only if the maximal rank of B_1 is 1.

(b) $a_{13}+a_{23}\equiv 0$ on U: In this case, f contains a constant light-like vector.

(b1) If the maximal rank of B_1 is 2, *then f is not (even locally) the conformal Gauss map of some Willmore immersion.*

(b2) If the maximal rank of B_1 is 1, *f belongs to one of the following two cases:*

(i) f is the conformal Gauss map of some Willmore surface $y:U\setminus U_0\rightarrow S^{n+2}$, and y is conformal to a minimal surface in R^{n+2}.

(ii) f reduces to a conformally harmonic map into $SO^+(1,n+1)/SO^+(1,1)\times SO(n)\subset SO^+(1,n+3)/SO^+(1,3)\times SO(n)$ or into $SO(n+2)/SO(2)\times SO(n)\subset SO^+(1,n+3)/SO^+(1,3)\times SO(n)$. In this case f is not (even locally) the conformal Gauss map of a Willmore immersion.

Proof. Note that for a harmonic map to be conformally harmonic, one only needs

$$\langle\alpha_1'(\frac{\partial}{\partial z}),\alpha_1'(\frac{\partial}{\partial z})\rangle=tr\left(\left(\alpha_1'(\frac{\partial}{\partial z})\right)^t I_{1,3}\alpha_1'(\frac{\partial}{\partial z})\right)=0.$$

But this follows immediately from the form of B_1 we have assumed.

The proof of parts (a) and (b) is based on an evaluation of the third of the harmonic map equations. Writing this equation in terms of matrix entries we obtain:

$$\beta_{j\bar{z}} - \bar{a}_{12}\beta_j - \frac{\sqrt{2}}{2}(\bar{a}_{13} + i\bar{a}_{14})k_j - \sum_{l=1}^{n} \beta_l \bar{b}_{lj} = 0, \tag{28}$$

$$-\beta_{j\bar{z}} + \bar{a}_{12}\beta_j - \frac{\sqrt{2}}{2}(\bar{a}_{23} + i\bar{a}_{24})k_j + \sum_{l=1}^{n} \beta_l \bar{b}_{lj} = 0, \tag{29}$$

$$-k_{j\bar{z}} + \sqrt{2}(\bar{a}_{13} + \bar{a}_{23})\beta_j - i\bar{a}_{34}k_j + \sum_{j=1}^{n} k_l \bar{b}_{lj} = 0, \tag{30}$$

$$-ik_{j\bar{z}} + \sqrt{2}(\bar{a}_{14} + \bar{a}_{24})\beta_j + \bar{a}_{34}k_j + i\sum_{j=1}^{n} k_l \bar{b}_{lj} = 0, \quad j = 1, \cdots, n. \tag{31}$$

Adding the first two equations and adding the third equation to i-times the fourth equation yields

$$(a_{13} + a_{23} - i(a_{14} + a_{24}))\bar{k}_j = (a_{13} + a_{23} - i(a_{14} + a_{24}))\bar{\beta}_j = 0, \ j = 1, \cdots, n.$$

Since f is non-constant, not all the β_j and all the k_j vanish and we infer

$$a_{13} + a_{23} = i(a_{14} + a_{24}). \tag{32}$$

Set $F = (e_0, \hat{e}_0, e_1, e_2, \psi_1, \cdots, \psi_n)$, and $Y_0 = \frac{1}{\sqrt{2}}(e_0 - \hat{e}_0)$, $\hat{Y}_0 = \frac{1}{\sqrt{2}}(e_0 + \hat{e}_0)$. Then

$$Y_{0z} = \frac{1}{\sqrt{2}}(e_0 - \hat{e}_0)_z = -a_{12}Y_0 + \frac{1}{\sqrt{2}}(a_{13} + a_{23})(e_1 - ie_2)$$

follows. Now there are two possibilities:

(a) $a_{13} + a_{23}$ does not vanish identically on U. Since $a_{13} + a_{23}$ is real analytic, there exists some subset U_0 of U satisfying the first part of the claim.

In this case one verifies directly that $[Y_0]$ is a conformal immersion into Q^{n+2}. Let $U_1 \subset U \setminus U_0$ be a simply connected subset. Then on U_1 we can write $a_{13} + a_{23} = \frac{1}{\sqrt{2}}re^{i\theta}$. Setting $Y_{01} = \frac{1}{r}Y_0$, $\hat{e}_1 + i\hat{e}_2 = e^{i\theta}(e_1 + e_2)$, we see that for corresponding the new frame $\hat{F}$ we obtain $\hat{a}_{13} + \hat{a}_{23} = \frac{1}{\sqrt{2}}$. So we can assume w.l.g. $a_{13} + a_{23} = \frac{1}{\sqrt{2}}$. Calculating

$$Y_{0z\bar{z}} \in \mathrm{Span}\left\{Y_0, \hat{Y}_0, e_1, e_2\right\}$$

shows that f is the harmonic conformal Gauss map of $[Y_0]$ on U_1. As a consequence, $[Y_0]$ is a Willmore surface on U_1. By real analyticity, $[Y_0]$ is Willmore on $U\setminus U_0$. The claim on S-Willmore surfaces comes from Corollary 2.10 and the fact that the rank of B_1 is independent of the choice of F.

As to the equivalence between f and the conformal Gauss map of $[Y_0]$ on the points of U_0, by the theorem of Hélein on the removability of singularities [35], the conformal Gauss map can be well-defined on such points and coincides with f. And since $[Y_0]$ is real analytic, it is also well-defined on U_0 and fails to be an immersion precisely on U_0.

(b). If $a_{13} + a_{23} = 0$ on U, then we obtain $Y_{0z} = -a_{12}Y_0$. By scaling, we may assume that $Y_{0z} = 0$ holds. (Hence we can assume w.l.g. $a_{12} = 0$ on U)

Consider an arbitrary null vector, except Y_0

$$Y_\mu = \hat{Y}_0 + \mu_1 e_1 - \mu_2 e_2 + \frac{|\mu|^2}{2} Y_0 = \hat{Y}_0 + \bar{\mu} P + \mu \bar{P} + \frac{|\mu|^2}{2} Y_0, \tag{33}$$

in the subspace $\mathrm{Span}_{\mathbb{R}}\{Y_0, \hat{Y}_0, e_1, e_2\}$, with $\mu = \mu_1 + i\mu_2$ a complex valued function and, $P = \frac{1}{2}(e_1 - ie_2)$. So $[Y_\mu]$ is a Willmore surface with the conformal Gauss map being f if and only if there exists some function μ such that

$$Y_\mu,\ Y_{\mu z},\ Y_{\mu z\bar{z}} \in \mathrm{Span}_{\mathbb{C}}\{e_0, \hat{e}_0, e_1, e_2\},\ \langle Y_{\mu z}, Y_{\mu z}\rangle = 0,\ \langle Y_{\mu z}, Y_{\mu \bar{z}}\rangle > 0. \tag{34}$$

(b1): Using $dF = F\alpha$ with α as in the assumption of the theorem we obtain

$$Y_{\mu z} = \sum_j^n (2\beta_j + \bar{\mu} k_j)\psi_j \mod \{e_0, \hat{e}_0, e_1, e_2\},$$

whence $Y_{\mu z} = 0 \mod \{e_0, \hat{e}_0, e_1, e_2\}$ implies $\beta_j = -\frac{\bar{\mu}}{2} k_j,\ j = 1, 2, \cdots, n$. From this we infer rank $B_1 \leq 1$ and (b1) follows.

(b2): Let's assume now that the maximal rank of B_1 is 1. We distinguish two cases. First we assume

Case $(b2.a)$: $\sum |k_j|^2 \neq 0$: Substituting $\beta_j = -\frac{\bar{\mu}}{2} k_j, j = 1, \ldots n$ into the equations (28) and using (32) , $a_{13} + a_{23} = 0$ and $a_{12} = 0$ we derive

$$\mu_z + \sqrt{2}(a_{13} - ia_{14}) + ia_{34}\mu = 0.$$

Differentiating (33) we observe

$$Y_{\mu z} = (\cdots)P + (\cdots)Y_\mu + (\mu_z + \sqrt{2}(a_{13} - ia_{14}) + ia_{34}\mu)\bar{P}, \tag{35}$$

whence $Y_{\mu z} = (\cdots)P + (\cdots)Y_0$ follows.

As a consequence we also obtain $\langle Y_{\mu z}, Y_{\mu z}\rangle = 0$. Moreover, using

$$P_{\bar{z}} = -i\bar{a}_{34} P + \frac{1}{\sqrt{2}}(\bar{a}_{13} - i\bar{a}_{14})Y_0,$$

one derives

$$Y_{\mu z\bar{z}} \in \mathrm{Span}\{e_0, \hat{e}_0, e_1, e_2\}.$$

At this point four of the five conditions listed in (34) are satisfied and Y_μ is a Willmore immersion if and only if the inequality $\langle Y_{\mu z}, Y_{\mu\bar{z}}\rangle > 0$ is satisfied.

(i). So on the open dense subset of U where $\langle Y_{\mu z}, Y_{\mu\bar{z}}\rangle > 0$, f is the conformally harmonic Gauss map of the conformal immersion $[Y_\mu]$ and $[Y_\mu]$ is S-Willmore. And since f contains a light-like vector Y_0, by the spherical projection with respect to Y_0, $[Y_\mu]$ becomes a minimal surface in R^{n+2} ([8], [31], [51], [50]).

(ii). When $\langle Y_{\mu z}, Y_{\mu\bar{z}}\rangle = 0$ on U, Y_μ is another constant light-like vector of f. Hence f reduces to a harmonic map into $SO(n+2)/SO(2) \times SO(n) \subset SO^+(1, n+3)/SO^+(1, 3) \times SO(n)$.

Case $(b2.b)$: $\sum |k_j|^2 \equiv 0$: In this case we have

$$e_{1z} = \sqrt{2}a_{13}Y_0 - a_{34}e_2,\ e_{2z} = \sqrt{2}a_{14}Y_0 + a_{34}e_1.$$

Therefore, by rotation of e_1, e_2, we may assume w.l.g. $a_{34} = 0$. See Lemma 4.2 in [60] for example. Hence, we can find some real functions $\tilde{\mu}_1, \tilde{\mu}_2$ such that

$$(e_1 + \tilde{\mu}_1 Y_0)_z = (e_2 + \tilde{\mu}_2 Y_0)_z = 0,$$

i.e.

$$e_1 + \tilde{\mu}_1 Y_0 = constant,\ e_2 + \tilde{\mu}_2 Y_0 = constant.$$

That is, f reduces to a map into $SO^+(1,n+1)/SO^+(1,1) \times SO(n) \subset SO^+(1,n+3)/SO^+(1,3) \times SO(n)$. $\square$

Remark 3.11. Note that the proof of Theorem 3.10 already provides a way to derive Willmore surfaces from the conformal Gauss maps.

Corollary 3.12. *Let f be a conformally harmonic map as in Theorem 3.10 which belongs to Case (a) as well as to Case (b). Then f is the conformal Gauss map of some minimal surface in R^{n+2} (after putting R^{n+2} conformally into S^{n+2}), and vice versa.*

Corollary 3.13. *Let f be a conformally harmonic map as in Theorem 3.10. Assume that f does not contain any constant lightlike vector.*

(a) If $rank B_1 = 2$, then there exists a unique (non-S-Willmore) Willmore surface $y : U \backslash U_0 \rightarrow S^{n+2}$ which has f as its conformal Gauss map.

(b) If $rank B_1 = 1$, then there exists a pair of dual S-Willmore surfaces $y, \hat{y} : U \backslash U_0 \rightarrow S^{n+2}$ where both have f as their conformal Gauss map.

Ejiri's Willmore torus in S^5 ([30]) provides a standard example for Case (a), and Veronese spheres in S^{2m} ([51]) provide examples for Case (b).

Corollary 3.14. *Let f be a conformally harmonic map as in Theorem 3.10. Assume that f does contain a constant lightlike vector. Then either*

a) f does not correspond to any immersion,

or

b) f corresponds to a Willmore immersion which is conformally equivalent to a minimal immersion into $\mathbb{R}^{n+2}$.

Remark 3.15. Theorem 3.10 shows that the case of f containing a constant light-like vector yields several possibilities. It actually takes a detailed study to find out in which of the cases the conformally harmonic map f does correspond to a Willmore surfaces.

In Section 5, we will give, in terms of normalized potentials, a simple description of those conformally harmonic maps which contain a constant lightlike vector. Since all the Willmore surfaces with conformal Gauss map containing a constant lightlike vector are known, for the construction of new Willmore surfaces we will only consider normalized potentials which do not satisfy the conditions defining those special harmonic maps.

In particular, we will obtain new Willmore spheres which are not S-Willmore if we assume that B_1 is of rank 2 and the harmonic map is a map from S^2 to S^{n+2}. (Such maps then actually are of finite uniton type, as we will show later.)

Theorem 3.16. *Let M be a connected Riemann surface and $\mathcal{U} = (U_j)_{j \in J}$ a cover of M consisting of open and contractible subsets of M. Let $f : M \rightarrow SO^+(1,n+3)/SO^+(1,3) \times SO(n)$ be a non-constant strongly conformally harmonic map. For $j \in J$, let f_j denote the restriction of f to U_j and let U_{j0} denote the points of U_j where f_j is not an immersion.*

a) If f_j is induced by some Willmore surface y_j on $U_j \setminus U_{j0}$ for some $j \in J$, then then all $f_k, k \in J$, are induced by some Willmore surface y_k on $U_k \setminus U_{k0}$.

b) In the case that f is locally induced by a Willmore immersion, on M or the double covering M^* of M, one can choose the Willmore maps $y_k, k \in J$, such that they are restrictions of a global Willmore map y.

Proof. a) Let's fix $j \in J$ and choose for $k \in J$ some point $p_k \in U_k$ and some curve γ_k from p_j to p_k. We can assume that there exists a simply connected open subset W_k containing the curve γ_k. We can extend the frame F_j along W_k and can thus produce a frame F_k on U_k. Actually, the proof of lemma 3.4 shows that we can assume w.l.g. that the first vector in the submatrix B_1 of the Maurer-Cartan form of F is one of the two special vectors $(1,-1,0,0,)^T$ or $(0,0,1,i)^T$.

Let's now consider the case, where F_j satisfies (a) of the last theorem. Then $b = a_{13} + a_{23}$ does not vanish identically on U_j, hence b does not vanish identically on W_k and therefore F_k also is in case (a). As a consequence, f is induced over U_k by a Willmore immersion, away from some subset $U_k \setminus U_{k0}$.

Analogously, if b is in the case (b) of the last theorem, then $b = 0$ on U_j, hence $b = 0$ on W_k and therefore $b = 0$ on U_k. Moreover, if the maximal rank of B_1 is 1 on U_1, then it is also 1 on W_k and on U_k. The proof of the last theorem shows that the case that the harmonic map f is induced by some Willmore surface (up to a singular set) is the one where $\sum \mid k_m \mid^2$ does not vanish identically. But this property again persists, starting from U_j, through W_k to U_k.

b) Let U_1 and U_2 be two sets of our open covering and assume $U_1 \cap U_2 \neq \emptyset$. Again we distinguish two cases.

Case 1: $rankB_1 = rankB_1' = 2$ on $(U_1 \cap U_2)\backslash U_0$, where U_0 is a nowhere dense subset of $U_1 \cap U_2$. Let $[Y_j]$ be the Willmore surfaces on U_j, $j = 1, 2$. By Theorem 2.13, $[Y_1] = [Y_2]$ on $(U_1 \cap U_2)\backslash U_0$, and the real analyticity of $[Y_1]$ and $[Y_2]$ shows $[Y_1] = [Y_2]$ on $(U_1 \cap U_2)$. Therefore we obtain a globally defined Willmore map on M.

Case 2: $rankB_1 = rankB_1' = 1$ on $(U_1 \cap U_2)$. By the proof of Theorem 2.13, we can consider in this case the possibly singular bundle $\hat{V} = Span\{Y_0, Y_\mu\}$ over M. Let M_1 be the subset of M such that $[Y_0] \neq [Y_\mu]$, which is open and dense in M. Let $\hat{V}_1$ denote the restriction of the bundle $\hat{V}$ on M_1. If $\hat{V}_1$ is orientable, noticing that there exist exactly two lightlike directions at $\hat{V}_1|_p$ for every $p \in M_1$, one will obtain a unique pair of lightlike vectors $\{Y_0, Y_\mu\}$ such that the first coordinates of Y_0, Y_μ are 1 and $\{Y_0, Y_\mu\}$ gives an orientation. Therefore one derives a globally defined $y = [Y_0]$ over M.

If $\hat{V}_1$ is non-orientable, one can use the double covering $(\hat{V}^*, M^*)$ of $(\hat{V}, M)$ such that the new bundle $\hat{V}_1^*$ over M_1^* is oriented. Hence the theorem follows.

It now suffices to prove the existence of the double covering $(\hat{V}^*, M^*)$ of $(\hat{V}, M)$. The proof is almost verbatim the same as for general manifolds (see e.g. page 105 of [37]). Let $\hat{\pi} : \hat{V} \to M$ denote the canonical projection. Since the Willmore maps are real analytic, on every U_j there exists a unique pair of lightlike vectors $\{Y_0, Y_\mu\}$ such that the first coordinates of Y_0, Y_μ are both 1. Hence $\{Y_0, Y_\mu\}$ and $\{Y_\mu, Y_0\}$ give two orientations over $U_j \cap M_1$, called ϖ and $-\varpi$ respectively. This also gives an orientation on U_j. Let

$$M^* = \Big\{(p, \varpi_p) \mid p \in M, \varpi_p \text{ is an orientation of } \hat{V}|_p \Big\}. \tag{36}$$

Let the topology of M^* be generated by the subsets

$$\{(p, \varpi_p) \mid p \in U_j, \varpi_p \text{ is an orientation of } \hat{V}^*|_{U_j}\}. \tag{37}$$

and define the natural projection map from M^* to M by $\pi_0(p, \pm\varpi) = p$. Set $Y^* = \pi_0^* Y$, for $Y \in Span\{Y_0, Y_\mu\}$. Then we obtain a double covering which we denote by $(\hat{V}^*, M^*)$. The bundle $\hat{V}^*$ carries a natural orientation. It is defined as follows: for any $(p, \varpi_p) \in M^*$, define the orientation of $\hat{V}^*$ at (p, ϖ_p) as the pullback by π_1^* of the orientation given by ϖ_p on $\hat{V}|_p$. $\square$

4 Loop group theory for harmonic maps

In this section we start by collecting the basic definitions and the basic decomposition theorems for loop groups ([26], [63], [3]). Next we recall the DPW method for the construction of har-

monic maps. Since in this paper we are mainly interested in conformally harmonic maps, we characterize conformally harmonicity in terms of the normalized potential. In view of our goal of presenting a new Willmore sphere in S^6 we show that all Willmore spheres in S^{n+2} are of finite uniton type. For this we relate a harmonic map into a non-compact symmetric space with a harmonic map into a compact one. This permits to apply work of Burstall and Guest [12].

4.1 Loop groups and decomposition theorems

Let $G^{\mathbb{C}}$ be a connected complex Lie group and $\mathfrak{g}^{\mathbb{C}}$ its Lie algebra. Let τ denote the complex anti-holomorphic involution $g \to \bar{g}$, of $G^{\mathbb{C}}$ and G its fixed point set. Then G is a real Lie group (not connected in our case). Its Lie algebra will be denoted by $\mathfrak{g}$. Suppose that $\sigma : G^{\mathbb{C}} \to G^{\mathbb{C}}$ is an inner involution which commutes with the complex conjugation τ. Let $K = \mathrm{Fix}^{\sigma}(G)$ and $K^{\mathbb{C}} = \mathrm{Fix}^{\sigma}(G^{\mathbb{C}})$ denote the corresponding fixed point groups. Note that $K^{\mathbb{C}}$ is connected, but K is not connected. Suppose that $K^{\mathbb{C}} = K \cdot B$ is an Iwasawa decomposition of $K^{\mathbb{C}}$ with B a maximal solvable subgroup such that $K \cap B = \{e\}$. Here e is the identity element of G. We define the twisted loop groups of G and $G^{\mathbb{C}}$ as follows:

$$\begin{aligned}
\Lambda G^{\mathbb{C}}_{\sigma} &= \{\gamma : S^1 \to G^{\mathbb{C}} \mid ,\ \sigma\gamma(\lambda) = \gamma(-\lambda), \lambda \in S^1\},\\
\Lambda G_{\sigma} &= \{\gamma \in \Lambda G^{\mathbb{C}}_{\sigma} | \ \gamma(\lambda) \in G, \text{for all } \lambda \in S^1\},\\
\Omega G_{\sigma} &= \{\gamma \in \Lambda G_{\sigma} | \ \gamma(1) = e\},\\
\Lambda^{-}_{*} G^{\mathbb{C}}_{\sigma} &= \{\gamma \in \Lambda G^{\mathbb{C}}_{\sigma} \mid \gamma \text{ extends holomorphically to } D_{\infty},\ \gamma(\infty) = e\},\\
\Lambda^{+} G^{\mathbb{C}}_{\sigma} &= \{\gamma \in \Lambda G^{\mathbb{C}}_{\sigma} \mid \gamma \text{ extends holomorphically to } D_0\},\\
\Lambda^{+}_{B} G^{\mathbb{C}}_{\sigma} &= \{\gamma \in \Lambda G^{\mathbb{C}}_{\sigma} \mid \gamma(0) \in B\},
\end{aligned}$$

where $D_0 = \{z \in \mathbb{C} | \ |z| < 1\}$, $D_{\infty} = \{z \in \mathbb{C} | \ |z| > 1\}$, and $\hat{\mathbb{C}}$ is the extended complex plane.

For the decomposition theorems quoted below we need to have some topology on our loop groups. This can be done in several ways. We will assume that all matrix entries are in the Wiener algebra of the unit circle. We thus obtain Banach Lie groups.

Theorem 4.1. *([26],[55])*

(i) (Iwasawa decomposition)

The multiplication $\Lambda G_{\sigma} \times \Lambda^{+}_{B} G^{\mathbb{C}}_{\sigma} \to \Lambda G^{\mathbb{C}}_{\sigma}$ is a diffeomorphism onto $\Lambda G^{\mathbb{C}}_{\sigma}$ when G is compact.

The multiplication $\Lambda G_{\sigma} \times \Lambda^{+}_{B} G^{\mathbb{C}}_{\sigma} \to \Lambda G^{\mathbb{C}}_{\sigma}$ is a diffeomorphism onto an open subset $\mathcal{I}^{\mathcal{U}} \subset \Lambda G^{\mathbb{C}}_{\sigma}$ when G is non-compact.

(ii) (Birkhoff decomposition)

The multiplication $\Lambda^{-}_{*} G^{\mathbb{C}}_{\sigma} \times \Lambda^{+} G^{\mathbb{C}}_{\sigma} \to \Lambda G^{\mathbb{C}}_{\sigma}$ is a diffeomorphism onto an open and dense subset $\Lambda^{-}_{*} G^{\mathbb{C}}_{\sigma} \cdot \Lambda^{+} G^{\mathbb{C}}_{\sigma}$ (big Birkhoff cell).

More information about the loop group decompositions theorems is contained in the appendix of this paper. One may also compare the decomposition theorems here with those stated in [3], [6], [42].

Loops which have a finite Fourier expansion will be called *algebraic loops* and will be denoted by the subscript "*alg*", like

$$\Lambda_{alg} G_{\sigma},\ \Lambda_{alg} G^{\mathbb{C}}_{\sigma},\ \Omega_{alg} G_{\sigma}$$

We define

$$\Omega^{k}_{alg} G_{\sigma} := \{\gamma \in \Omega_{alg} G_{\sigma} | Ad(\gamma) = \sum_{|j| \leq k} \lambda^{j} T_{j}\} \ .$$

For later purposes we define

Definition 4.2. *([57],[12]) (i) Let $f : M \to G$ be a harmonic map into a real Lie group G with extended solution $\Phi(z,\lambda) \in \Lambda G^{\mathbb{C}}_{\sigma}$. We say that f has finite uniton number k if*

$$\Phi(M) \subset \Omega^k_{alg} G_\sigma, \quad \text{and } \Phi(M) \nsubseteq \Omega^{k-1}_{alg} G_\sigma.$$

In this case we write $r(f) = k$.

(ii) Let G/K be an inner symmetric space (given by the inner involution $\sigma : G \to G$). We map G/K into G as totally geodesic submanifold via the (finite covering) Cartan map:

$$\begin{aligned} \mathcal{C} : G/K &\to G \\ gK &\mapsto g\sigma(g)^{-1}. \end{aligned} \tag{38}$$

A harmonic map f into G/K is said to be of finite uniton number k if it is of finite uniton number k when considered as a harmonic map into G via the Cartan map, i.e., f has finite uniton number k if and only if $\mathcal{C} \circ f$ has finite uniton number k. (See [12] for more details).

4.2 The DPW method and potentials

With the loop group decompositions as stated above, we obtain a construction scheme of harmonic maps from a surface into any real affine symmetric space G/K.

So far we have mainly discussed Willmore surfaces and the corresponding conformally harmonic maps defined on some open subset U of $\mathbb{C}$ (or possibly an open subset of some surface M). Since the immersions of interest are conformal, the corresponding surface has a complex structure. We thus only consider Riemann surfaces. If M is such a Riemann surface, then its universal cover $\tilde{M}$ is either S^2 or $\mathbb{C}$ or $\mathbb{E}$, the open unit disk in $\mathbb{C}$. Every harmonic map from M to some affine symmetric space G/K induces via composition with the natural projection a harmonic map from the universal cover $\tilde{M}$ into G/K. Therefore, to start with, we need to consider harmonic maps from S^2, $\mathbb{C}$ and $\mathbb{E}$ into G/K.

Theorem 4.3. *([26]) Let $\mathbb{D}$ be a contractible open subset of $\mathbb{C}$ and $z_0 \in \mathbb{D}$ a base point. Let $f : \mathbb{D} \to G/K$ be a harmonic map with $f(z_0) = eK$. Then the associated family f_λ of F can be lifted to a map $F : \mathbb{D} \to \Lambda G_\sigma$, the extended frame of f and we can assume w.l.g. that $F(z_0, \lambda) = e$ holds. Under this assumption,*

(1) The map F takes only values in $\mathcal{I}^{\mathcal{U}} \subset \Lambda G^{\mathbb{C}}_{\sigma}$.

(2) There exists a discrete subset $\mathbb{D}_0 \subset \mathbb{D}$ such that on $\mathbb{D} \setminus \mathbb{D}_0$ we have the decomposition

$$F(z,\lambda) = F_-(z,\lambda) \cdot F_+(z,\lambda),$$

where

$$F_-(z,\lambda) \in \Lambda^-_* G^{\mathbb{C}}_{\sigma} \ \text{ and } \ F_+(z,\lambda) \in \Lambda^+ G^{\mathbb{C}}_{\sigma}.$$

Moreover $F_-(z,\lambda)$ is meromorphic in $z \in \mathbb{D}$ and $F_-(z_0,\lambda) = e$ holds.

Moreover,

$$\eta = F_-(z,\lambda)^{-1} dF_-(z,\lambda)$$

is a $\lambda^{-1} \cdot \mathfrak{p}^{\mathbb{C}}-$ valued meromorphic $(1,0)-$ form with poles at points of $\mathbb{D}_0$ only.

(3) *Conversely, any harmonic map $f : \mathbb{D} \to G/K$ can be derived from a $\lambda^{-1} \cdot \mathfrak{p}^{\mathbb{C}}-$ valued meromorphic $(1,0)-$ form η on $\mathbb{D}$.*

(4) *Spelling out the converse procedure in detail we obtain: Let η be a $\lambda^{-1} \cdot \mathfrak{p}^{\mathbb{C}}-$ valued meromorphic $(1,0)-$ form for which the solution to the ODE*

$$F_-(z,\lambda)^{-1} dF_-(z,\lambda) = \eta, \quad F_-(z_0,\lambda) = e, \tag{39}$$

is meromorphic on $\mathbb{D}$, with $\mathbb{D}_0$ as set of possible poles. Then on the open set $\mathbb{D}_{\mathcal{I}} = \{z \in \mathbb{D}; F(z,\lambda) \in \mathcal{I}^{\mathcal{U}}\}$ we define $\tilde{F}(z,\lambda)$ via the factorization $\mathcal{I}^{\mathcal{U}} = (\Lambda G_\sigma)^0 \cdot \Lambda_B^+ G_\sigma^{\mathbb{C}} \subset \Lambda G_\sigma^{\mathbb{C}}$:

$$F_-(z,\lambda) = \tilde{F}(z,\lambda) \cdot \tilde{F}_+(z,\lambda)^{-1}. \tag{40}$$

This way one obtains an extended frame

$$\tilde{F}(z,\lambda) = F_-(z,\lambda) \cdot \tilde{F}_+(z,\lambda)$$

of some harmonic map from $\mathbb{D} \setminus \mathbb{D}_{\mathcal{I}}$ to G/K satisfying $\tilde{F}(z_0,\lambda) = e$.

Moreover, the two constructions outlined above are inverse to each other (on appropriate domains of definition).

Definition 4.4. *([26]) The $\lambda^{-1} \cdot \mathfrak{p}^{\mathbb{C}}-$ valued meromorphic $(1,0)$ form η is called the* normalized potential *for the harmonic map f with the point z_0 as the reference point.*

Remark 4.5. Note that the normalized potential is uniquely determined once a base point is chosen. However, if we conjugate a normalized potential by some element k of K, then the procedure outlined in the theorem produces a new harmonic map (and correspondingly a new Willmore surface) which differs from the original one by the rigid motion induced by k. Since we usually do not care about how the harmonic map (or the Willmore surface) sits in its space, we sometimes use elements of K to simplify or further normalize the normalized potential.

Remark 4.6. In the converse procedure, part (4) above, since in our case the symmetric space G/K is not compact, the Iwasawa splitting (40) will in general not be possible for all $z \in \mathbb{D}$. Thus $\tilde{F}$, as well as the harmonic map $\tilde{f}$ will have singularities on $\mathbb{D}$. There are two types of singularities. One type stems from poles in the potential η and the other type occurs when F_- crosses from one open Iwasawa cell into the other one. (There are two open Iwasawa cells, as is shown in the appendix.) In our new example of a Willmore sphere in S^6 it happens that the frame has a singularity, but the harmonic map and the Willmore immersion do not have any singularity, since the projection of the frame into the harmonic map deletes the singular denominator, and the singular denominator cancels also in the transition from the harmonic map to the projectivized light cone, the sphere S^6.

Remark 4.7. So far we have only introduced the "normalized potential". However, in many applications it is much more convenient to use potentials which have in their Fourier expansion more than one power of λ occurring. The normalized potential is uniquely determined, if some base point on M is fixed and the frames are normalized to e at the base point. The normalized potential is usually meromorphic in z. And since it is uniquely determined, there is no way to change this. However, when permitting many (maybe infinitely many) powers of λ, then one can obtain holomorphic coefficients.

Theorem 4.8. *Let $\mathbb{D}$ be a contractible open subset of $\mathbb{C}$. Let $F(z,\lambda)$ be the frame of some harmonic map into G/K. Then there exists some $V_+ \in \Lambda^+ G_\sigma^{\mathbb{C}}$ such that $C(z,\lambda) = FV_+$ is holomorphic in z and in $\lambda \in \mathbb{C}^*$. Then the Maurer-Cartan form $\eta = C^{-1}dC$ of C is a holomorphic $(1,0)-$form on $\mathbb{D}$ and it is easy to verify that $\lambda\eta$ is holomorphic for $\lambda \in \mathbb{C}$. Conversely, any harmonic map $f : \mathbb{D} \to G/K$ can be derived from such a holomorphic $(1,0)-$ form η on $\mathbb{D}$ by the same steps as in the previous theorem. These two procedures are inverse to each other, if normalizations at some base point are used.*

The proof can be taken verbatim from the appendix of [26] and will therefore be omitted here.

Remark 4.9. Of course, in the converse procedure of the last theorem the Iwasawa splitting (40) will in general not be possible either for all $z \in \mathbb{D}$ since our symmetric space G/K is not compact (compare [6], [27]).

Remark 4.10. Let η_1 and η_2 be any two potentials producing the same harmonic map by the procedure outlined above. Then there exists a gauge $W_+ : \mathbb{D} \to \Lambda^+ G_\sigma^{\mathbb{C}}$ transforming one potential into the other. For a proof consider the frames $F_1 = C_1 V_{+1}$ and $F_2 = C_2 V_{+2}$ constructed as outlined above. Since we assume that the two potentials induce the same harmonic map, these frames only differ by some gauge: $F_1 = F_2 T$ where $T \in K$. This implies $C_1 V_{+1} = C_2 V_{+2} T$. Thus $W_+ = V_{+2} T V_{+1}^{-1}$ is the desired gauge.

So far we have only discussed potentials for harmonic maps defined on some contractible open subset of $\mathbb{C}$. Let now M denote a Riemann surface which is either non-compact or compact of positive genus. Then the universal cover $\tilde{M}$ of M can be realized as a contractible open subset of $\mathbb{C}$. Moreover, if $f : M \to G/K$ is a harmonic map, then the composition $\tilde{f}$ of f with the canonical projection from $\tilde{M}$ onto M is also harmonic. Therefore to $\tilde{f}$ we can construct normalized potentials and holomorphic potentials as outlined above. These potentials for $\tilde{f}$ will also be called potentials for f. The converse procedure as outlined in the last two theorems produces harmonic maps defined on some open subsets (containing the base point) of $\mathbb{D}$. For these harmonic maps to descend to M "closing conditions" need to be satisfied.

The following result can be proven as in [26], [20], [21], [22].

Theorem 4.11. *Let M be a Riemann surface which is either non-compact or compact of positive genus.*

(1) Let $f : M \to G/K$ be a harmonic map and $\tilde{f}$ its lift to the universal cover $\tilde{M}$. Then here exists a normalized potential and a holomorphic potential for f, namely the corresponding potentials for $\tilde{f}$.

(2) Conversely, starting from some potential producing a harmonic map $\tilde{f}$ from $\tilde{M}$ to G/K, one obtains a harmonic map f on M if and only if

(2a) The monodromy matrices $\chi(\gamma, \lambda)$ associated with $\gamma \in \pi_1(M)$, considered as automorphisms of $\tilde{M}$, are elements of $(\Lambda G_\sigma)^0$.

(2b) There exists some $\lambda_0 \in S^1$ such that

$$\begin{aligned} F(\gamma \cdot z, \overline{\gamma \cdot z}, \lambda = \lambda_0) &= \chi(\gamma, \lambda = \lambda_0) F(z, \bar{z}, \lambda = \lambda_0) \\ &= F(z, \bar{z}, \lambda = \lambda_0) \quad \mod K \end{aligned} \tag{41}$$

for all $\gamma \in \pi_1(M)$.

For more details of this theorem as well as the definition of monodromy etc., we refer to [20], [21], [22].

Remark 4.12. (i). For the sphere $M = S^2$ the procedure discussed above works just as well if the symmetric space actually is a real Lie group G considered as a symmetric space $G \cong (G \times G)/\Delta$, where Δ denotes the subgroup $\Delta = \{(g, g) \in G \times G, g \in G\}$ and one uses the natural projection $(g, h) \to gh^{-1}$. Then on can lift a harmonic map $f : S^2 \to G$ to $G \times G$ by $F = (f, e)$. This way one obtains, as in the previous cases, a normalized potential of the form $\xi = (\lambda^{-1}\eta, -\lambda^{-1}\eta)$. Harmonic maps into Lie groups (as symmetric spaces) have been discussed in [18], Section 9 (also see the discussion below). Note, however, that the formula given in [18] for the normalized potential shows a wrong $\lambda-$ dependence.

(ii). On the other hand, one does not obtain a holomorphic potential for $M = S^2$, since S^2 does not carry any non-trivial holomorphic $(1,0)-$forms. The proof of [26] which was applied above is not applicable in the case $M = S^2$, of course.

Let's consider now the case that we have a harmonic map f from $M = S^2$ into some symmetric space G/K. In this case it may not be possible to lift f to a map F from S^2 to G such that the map F composed with the natural projection from G to G/K is f ([10]). But one can find some way around this non-lifting obstacle and derive that

Theorem 4.13. *Every harmonic map from S^2 to any inner symmetric space G/K can be obtained from some meromorphic normalized potential.*

Proof. Consider the smooth Cartan map $g \to g\sigma(g)^{-1}$ already used in (38) from G/K to G ([12], [18]). This map is for all inner symmetric spaces a totally geodesic, isometric finite covering of the connected component of the submanifold $S_\sigma = \{g \in G, \sigma(g) = g^{-1}\}$ of G which contains e. As a consequence, the composition of these maps is harmonic as well and we can apply what we discussed above. We can thus define a harmonic map $M = S^2 \to G \times G$ by the formula

$$p \in S^2 \to f(p) \in G/K \to g\sigma(g)^{-1} \in G \to (g\sigma(g)^{-1}, e) \in G \times G. \tag{42}$$

The map $f(p) \to g = g(p)$ is determined only up to some multiplication by some element of K from the right. Therefore we can consider just as well

$$\tilde{f}(p) = (g, \sigma(g)) \tag{43}$$

We know that at least in a small neighborhood of any point p we can choose g to be real analytic. Therefore, locally around every point p of S^2 we can assume that g is a smooth frame for f in G. In particular, g satisfies the usual Lax pair equations for frames of harmonic maps and one can introduce the parameter λ and extend g locally to an extended frame of f. This implies that we can consider the frame $g(z, \bar{z}, \lambda)$ as an element in the twisted loop group $(\Lambda G_\sigma)^0$. Moreover, we have the generalized Cartan immersion for every $\lambda \in \mathbb{C}^*$, whence the formula (43) holds (locally in p) for every $\lambda \in \mathbb{C}^*$.

Let's consider now the loop group procedure as presented in [26], some of which is also contained in [18]. Since the involutory automorphism of $G \times G$ defining the symmetric space G is just the interchanging map ρ, it is easy to verify that the twisted "double loop group" is

$$\Lambda(G \times G)_\rho = \{(g(\lambda), g(-\lambda)), g(\lambda) \in \Lambda G\}. \tag{44}$$

Correspondingly we obtain

$$\Lambda^+(G \times G)^{\mathbb{C}}_\rho = \{(g(\lambda), g(-\lambda)), g(\lambda) \in \Lambda^+ G^{\mathbb{C}}\}. \tag{45}$$

Similarly we obtain the description of $\Lambda^-(G \times G)^{\mathbb{C}}_\rho$. Note that the matrices occurring here are not $\sigma-$twisted. The $\sigma-$twisting is encoded in the different sign of λ in the two factors. It is a fortunate coincidence, that a matrix $g \in (\Lambda G)^0$ is in the twisted loop group $(\Lambda G_\sigma)^0$ if and only if the pair $(g(\lambda), \sigma(g(\lambda)))$ is in the $\rho-$twisted loop group associated with $G \times G$ (Here the superscript " ^{0}" denotes the identity component). This has the consequence that the map (43) is actually a map into $\Lambda(G \times G)_\rho$. In other words, we have a local lift $F(z, \bar{z}, \lambda)$ for every $\lambda \in \mathbb{C}^*$

$$F(z, \bar{z}, \lambda) = (g(z, \bar{z}, \lambda), \sigma(g(z, \bar{z}, \lambda))) \in \Lambda(G \times G)^{\mathbb{C}}_\rho. \tag{46}$$

If $\tilde{g}$ is another local lift, then on the domain of intersection g and $\tilde{g}$ only differ by some gauge from K: $\tilde{g}(p) = g(p)k(p)$. Then $\tilde{g}(z, \bar{z}, \lambda) = g(z, \bar{z}, \lambda)k(z, \bar{z})$ and

$$\tilde{F}(z, \bar{z}, \lambda) = F(z, \bar{z}, \lambda)\mathcal{K}(z, \bar{z}), \tag{47}$$

where $\mathcal{K} = (k, k)$. We can assume w.l.g. that $F(z_0, \bar{z}_0, \lambda) = I$ if z_0 is in the domain of definition of g.

The loop group method now requires to perform a Birkhoff decomposition

$$F(z,\bar{z},\lambda) = F_-(z,\bar{z},\lambda)F_+(z,\bar{z},\lambda) \tag{48}$$

with $F_\varepsilon(z,\bar{z},\lambda) \in \Lambda^\varepsilon(G\times G)^{\mathbb{C}}_\rho$. Note (47) implies that F_- is for every $p \in M$ independent of the choice of local lift g. Let's look at F_- in more detail. Writing out (48) we consider

$$(g(z,\bar{z},\lambda),\sigma(g(z,\bar{z},\lambda))) = (u_-(z,\lambda),u_-(z,-\lambda))\,(V_+(z,\bar{z},\lambda),V_+(z,\bar{z},-\lambda)) \tag{49}$$

with $u_- \in \Lambda_* G^{\mathbb{C}}$ and $V_+ \in \Lambda G^{\mathbb{C}}$. Whence

$$g(z,\bar{z},\lambda) = u_-(z,\lambda)V_+(z,\bar{z},\lambda), \quad \text{and} \quad \sigma\left(g(z,\bar{z},\lambda)\right) = u_-(z,-\lambda)V_+(z,\bar{z},-\lambda).$$

For our purposes we want $u_- \in \Lambda_* G^{\mathbb{C}}_\sigma$. Comparing the two equations above we obtain

$$u_-(z,\lambda)^{-1}\sigma(u_-(z,-\lambda)) = \sigma(V_+(z,\bar{z},-\lambda))V_+(z,\bar{z},\lambda)^{-1} = q(z,\bar{z},\lambda). \tag{50}$$

This implies that the expressions in (50) are independent of λ. But the left-hand expression in (50) has I as $\lambda-$independent term. Therefore $q(z,\bar{z},\lambda) = I$ and u_- and V_+ are twisted with respect to σ. Note that u_- is defined and meromorphic on S^2. We claim that $\eta_- = u_-^{-1}\frac{d}{dz}u_-$ ($\in \lambda^{-1}\mathfrak{g}^{\mathbb{C}}$) provides a normalized potential for f:

Starting from η_- we solve the ODE

$$u_-^{-1}du_- = \eta_- dz, \;\; u_-(z_0,\lambda) = I$$

with the u_- we already know. Considering next $F_-(z,\lambda) = (u_-(z,\lambda),\, u_-(z,-\lambda))$ we know that an Iwasawa splitting exists ($F_- = FF_+^{-1}$). Let $\check{F}$ denote the unique Iwasawa factor away from poles of u_-. Then locally $(g,\sigma(g)) = \check{F}\check{\mathcal{K}}$ and the corresponding harmonic map $\check{f}$ satisfies $\check{f} = \check{F}$ mod $Fix(\rho) = (g\sigma(g)^{-1},e) = (f,e)$.

□

Remark 4.14. Let $f : S^2 \to G/K$ be a harmonic map and η its normalized potential with reference point z_0. Then away from the (finitely many) poles of η there exists an extended frame F for f and a global Iwasawa splitting $F = F_-F_+$, $F_-^{-1}\frac{d}{dz}F_- = \eta$. Moreover, F_- is meromorphic on S^2.

Clearly, the normalized potential just discussed lives on $S^2 = M$. If one wants to construct harmonic maps from some arbitrary Riemann surface M into some affine symmetric space G/K, one has at least some indication for where to find an appropriate potential, if one knows that for every harmonic map from M to G/K there is some potential defined on M. So far there is known [24], Theorem 3.2

Theorem 4.15. *If M is non-compact, then for every harmonic map from M to any affine symmetric space there exists a holomorphic potential defined on M (more precisely, there exists a potential on the universal cover $\tilde{M}$ of M which is invariant under the fundamental group of M.)*

We conjecture more generally

Every harmonic map from any compact Riemann surface M to any affine symmetric space can be obtained from some meromorphic potential defined on M.

Remark 4.16. The statement that every harmonic map from S^2 to some affine symmetric space G/K can be obtained from some meromorphic potential on S^2 means more precisely: the ODE $dC = C\eta$ has a meromorphic solution and the "frame" obtained via Iwasawa splitting is rational in the variables z and $\bar{z}$. The projection onto G/K then yields a smooth harmonic map. (If one starts from some arbitrary potential, singularities may occur.)

The conjecture is that this works the same way if M is an arbitrary compact Riemann surface.

4.3 Wu's Formula

From the definition of the normalized potential we can read off that it is obtained from the $\lambda^{-1}-$part of the Maurer Cartan form of F by conjugation by some matrix function with values in $K^{\mathbb{C}}$. For known examples one can write down the normalized potential much more specifically. In [63], Wu showed how one can determine locally the normalized potential from the Maurer-Cartan form of the harmonic map f. Suppose that $\mathbb{D}$ is contractible and let's choose for simplicity w.l.g. the base point $z_0 = 0$. Let δ_1 denote the sum of the holomorphic terms in the Taylor expansion of α_1' about 0, considered as a form depending on z and $\bar{z}$. The form δ_1 is called the *holomorphic part* of α_1'. Similarly, denote by δ_0 the holomorphic part of α_0'. Then we obtain

Theorem 4.17. *(Wu's Formula [63]) Let $\mathbb{D}$ be a contractible open subset of $\mathbb{C}$ and $0 \in \mathbb{D}$ a base point. Let $f : \mathbb{D} \to G/K$ be a harmonic map with $f(0) = eK$. Then the normalized potential η of f with the origin as the reference point is given by*

$$\eta = F_0(z)\delta_1 F_0(z)^{-1}, \tag{51}$$

where $F_0 : \mathbb{D} \to G/K$ is the solution of the equation $F_0(z)^{-1}dF_0(z) = \delta_0$, $F_0(0) = F(0)$ and $F(0) \in K$ is the value of some frame for f at $z = 0$.

Note that one can actually show (similar to [25]) that the Maurer-Cartan form α of F has a meromorphic extension to $\mathbb{D} \times \mathbb{D}$, permitting to replace $\bar{z}$ globally by a free variable. As an application of Wu's formula, we consider the normalized potential of a conformally harmonic map into $SO^+(1, n+3)/SO^+(1,3) \times SO(n)$.

Hence we obtain the normalized potential by conjugation of the holomorphic part of α_1' of α by a map

$$F_0 = \mathrm{diag}\{\hat{A}_1, \hat{A}_2\} : M \to SO^+(1,3,\mathbb{C}) \times SO(n,\mathbb{C})$$

(which can be specified further).

By Theorem 3.16 we know that the $(1,0)-$part of α_1 has the form

$$\alpha_1' = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{pmatrix} dz.$$

Therefore, the holomorphic part of α_1' is of the form

$$\delta_1' = \begin{pmatrix} 0 & B_1' \\ -{B'}_1^t I_{1,3} & 0 \end{pmatrix} dz.$$

Then

$$\eta = \lambda^{-1} \cdot \begin{pmatrix} 0 & \hat{A}_1 B_1' \hat{A}_2^{-1} \\ -\hat{A}_2 {B'}_1^t I_{1,3} \hat{A}_1^{-1} & 0 \end{pmatrix} dz = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz.$$

Recall that B_1 satisfies $B_1^t I_{1,3} B_1 = 0$. Hence we have that ${B'}_1^t I_{1,3} {B'}_1 = 0$, showing that

$$\hat{B}_1{}^t I_{1,3} \hat{B}_1 = 0.$$

Theorem 4.18. *Let $\mathbb{D}$ be a contractible open subset of $\mathbb{C}$ and $0 \in \mathbb{D}$ a base point. Let $f : \mathbb{D} \to SO^+(1, n+3)/SO^+(1,3) \times SO(n)$ be a strongly conformally harmonic map with $f(0) = eK$ and $F : \mathbb{D} \to (\Lambda G_\sigma)^0$ an extended frame of f such that $F(0, \lambda) = I$. Then the normalized potential of f is of the form*

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \quad \textit{where} \quad \hat{B}_1{}^t I_{1,3} \hat{B}_1 = 0, \tag{52}$$

and where $\hat{B}_1 dz$ is a meromorphic $(1,0)-$ form on $\mathbb{D}$.

Conversely, any such normalized potential defined on $\mathbb{D}$ gives a strongly conformally harmonic map from an open subset $0 \in \mathbb{D}_{\mathcal{I}} \subset \mathbb{D}$ into $SO^+(1,n+3)/SO^+(1,3) \times SO(n)$.

Remark 4.19. It is straightforward to verify that

$$\begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix}^3 = 0.$$

So η is pointwise nilpotent as a Lie algebra-valued function. However this does not imply that η attains all values in a fixed nilpotent Lie subalgebra. As a consequence, in general the corresponding conformally harmonic map is not of finite uniton type. A standard example for this is the Clifford torus in S^3, which is of finite type and not of finite uniton type.

4.4 Harmonic maps into non-compact symmetric spaces and associated harmonic maps into their compact dual

The main interest in this paper is to discuss Willmore surfaces and thus to discuss conformally harmonic maps into a specific non-compact, inner symmetric space. In this subsection we assume more generally that G is connected, non-compact, semi-simple, real Lie group and G/K a non-compact inner symmetric space with inner involution σ on G. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}}$ its complexification. Then we obtain, as usual, three involutory, pairwise commuting automorphisms of $\mathfrak{g}^{\mathbb{C}}$: The complex linear extension of σ to $\mathfrak{g}^{\mathbb{C}}$, the complex antilinear involution, called τ, which defines the real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ and an involution θ which is the complex linear extension of a Cartan involution on $\mathfrak{g}$ and which commutes with σ.

Let $G^{\mathbb{C}}$ denote the simply connected complex Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Then σ, τ and θ have extensions to pairwise commuting group homomorphisms of $G^{\mathbb{C}}$.

Let $f: M \to G/K$ be a harmonic map with an extended frame $F: \tilde{M} \to \Lambda G_{\sigma} \subset \Lambda G_{\sigma}^{\mathbb{C}}$. We want to relate f to a harmonic map $\hat{f}$ into a compact inner symmetric space.

Let $U = Fix(\theta)$, then U is a maximal compact subgroup of $G^{\mathbb{C}}$ and U is simply connected ([1]). Moreover, observe that $Fix(\sigma) = K^{\mathbb{C}}$ is a connected complex Lie group (see [1], Lemma 2) with $K \subset K^{\mathbb{C}} \cap G$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ relative to σ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the decomposition of $\mathfrak{g}$ relative to θ. Then

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{k} \cap \mathfrak{m} + \mathfrak{p} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{m}$$

as a direct sum of vector spaces. Moreover, for the Lie algebra $\mathfrak{u}$ of U we have

$$\begin{aligned} \mathfrak{u} &= \mathfrak{k} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{h} + i\,(\mathfrak{k} \cap \mathfrak{m} + \mathfrak{p} \cap \mathfrak{m}) \\ &= (\mathfrak{k} \cap \mathfrak{h} + (i\mathfrak{k}) \cap (i\mathfrak{m})) + (\mathfrak{p} \cap \mathfrak{h} + (i\mathfrak{p}) \cap (i\mathfrak{m})) \\ &= \mathfrak{k}^{\mathbb{C}} \cap \mathfrak{u} + \mathfrak{p}^{\mathbb{C}} \cap \mathfrak{u}. \end{aligned}$$

It is easy to see now that $\left(\mathfrak{k}^{\mathbb{C}} \cap \mathfrak{u}\right)^{\mathbb{C}} = (\mathfrak{k} \cap \mathfrak{h} + (i\mathfrak{k}) \cap (i\mathfrak{m}))^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}}$ holds.

As a consequence, for the maximal compact Lie subgroup U of $G^{\mathbb{C}}$ constructed above we obtain

$$(U \cap K^{\mathbb{C}})^{\mathbb{C}} = K^{\mathbb{C}}.$$

It is not hard to prove that (the proof is postponed to Appendix B)

Lemma 4.20. *The symmetric space $(U \cap K^{\mathbb{C}})^{\mathbb{C}} = K^{\mathbb{C}}$ is an inner symmetric space.*

Example 4.21. For the strongly conformally harmonic map associated with a strong Willmore map $f : M \to S^{n+2}$ we have

$$G = SO^+(1, n+3), \quad G^{\mathbb{C}} = SO(1, n+3, \mathbb{C}), \quad \text{and} \quad K = SO(1,3) \times SO(n).$$

Hence $U \cong SO(n+4)$, and $U \cap K^{\mathbb{C}} = SO(4) \times SO(n)$. Moreover,

$$Lie(U) = \mathfrak{u} = \{A \in \mathfrak{so}(1, n+3, \mathbb{C}) | A = (a_{jk}),\ ia_{1j} \in \mathbb{R}, j = 1, \cdots, n+4\},$$

$$(\mathfrak{u} \cap \mathfrak{k}^{\mathbb{C}})^{\mathbb{C}} = \mathfrak{so}(1,3,\mathbb{C}) \times \mathfrak{so}(n, \mathbb{C}) = \mathfrak{k}^{\mathbb{C}}.$$

Now we are in the position to prove

Theorem 4.22. *Let $f : \tilde{M} \to G/K$ be a harmonic map from a simply connected Riemann surface $\tilde{M}$ into an inner, non-compact, symmetric space G/K and assume that its extended frame F only attains values in the big cell $\mathcal{I}^{\mathcal{U}} \subset \Lambda G^{\mathbb{C}}_{\sigma}$. Let U be a maximal compact Lie subgroup of $G^{\mathbb{C}}$ satisfying $(U \cap K^{\mathbb{C}})^{\mathbb{C}} = K^{\mathbb{C}}$. Then there exists a (new) harmonic map $f_U : \tilde{M} \to U/(U \cap K^{\mathbb{C}})$ into the compact symmetric space $U/(U \cap K^{\mathbb{C}})$ which has the same normalized potential as f. The map f_U is induced from f via the Iwasawa decomposition of F relative to U.*

Proof. So far for the harmonic map f, we have considered the Iwasawa decomposition in $\Lambda G^{\mathbb{C}}_{\sigma}$ relative to ΛG_{σ}. Restricting σ to U, we can consider the twisted loop group $\Lambda U^{\mathbb{C}}_{\sigma}$ and the corresponding Iwasawa decomposition of $\Lambda U^{\mathbb{C}}_{\sigma}$ relative to ΛU_{σ}. By our construction, complexifying G and complexifying U yields the same complex Lie group $G^{\mathbb{C}}$ and the holomorphic extensions of σ from G and from U yield the same involution of $G^{\mathbb{C}}$. Therefore the complex twisted loop groups, constructed starting with G and starting with U are the same, that is,

$$\Lambda G^{\mathbb{C}}_{\sigma} = \Lambda U^{\mathbb{C}}_{\sigma}, \quad \text{and} \quad \Lambda^+ G^{\mathbb{C}}_{\sigma} = \Lambda^+ U^{\mathbb{C}}_{\sigma}.$$

Applying this we can perform another Iwasawa decomposition of $\Lambda G^{\mathbb{C}}_{\sigma}$:

$$\Lambda U_{\sigma} \cdot \Lambda^+ G^{\mathbb{C}}_{\sigma} = \Lambda G^{\mathbb{C}}_{\sigma}.$$

Now let us turn to the harmonic map. First we assume that $\tilde{M} = \mathbb{D}$ is a contractible open subset of $\mathbb{C}$. Then we derive a global extended frame $F(z, \bar{z}, \lambda)$ of f. Applying this decomposition to the frame F we obtain

$$F = F_U \cdot W_+, \quad F_U \in \Lambda U_{\sigma}, \quad W_+ \in \Lambda^+ U^{\mathbb{C}}_{\sigma}. \tag{53}$$

Writing as usual $\alpha = F^{-1} dF = \lambda^{-1} \alpha'_{\mathfrak{p}} + \alpha_0 + \lambda \alpha''_{\mathfrak{p}}$, we obtain

$$F_U^{-1} dF_U = \alpha_U = W_+ \alpha W_+^{-1} - dW_+ W_+^{-1} = \lambda^{-1} W_0 \alpha'_{\mathfrak{p}} W_0^{-1} + \alpha_{U,0} + \lambda \alpha_{U,1} \ldots.$$

Since $W_+ \in \Lambda^+ U^{\mathbb{C}}_{\sigma} = \Lambda^+ G^{\mathbb{C}}_{\sigma}$, we have

$$W_0 \in K^{\mathbb{C}}, \quad \text{and} \quad \sigma(W_0) = W_0.$$

Since $\alpha'_{\mathfrak{p}} \in \Lambda \mathfrak{p}^{\mathbb{C}}_{\sigma}$ we obtain moreover

$$\sigma(\alpha'_{\mathfrak{p}}) = -\alpha'_{\mathfrak{p}}, \quad \text{and} \quad \sigma(W_0 \alpha'_{\mathfrak{p}} W_0^{-1}) = -W_0 \alpha'_{\mathfrak{p}} W_0^{-1}.$$

Since α_U is fixed by the anti-holomorphic involution θ we infer

$$\alpha_U = \lambda^{-1} \hat{\alpha}'_{\mathfrak{p}} + \hat{\alpha}_{\mathfrak{k}} + \lambda \hat{\alpha}''_{\mathfrak{p}},$$

where

$$\hat{\alpha}_{\mathfrak{k}} \in \mathfrak{u} \cap \mathfrak{k}^{\mathbb{C}}, \text{ and } \hat{\alpha}''_{\mathfrak{p}} = \theta\left(\lambda \hat{\alpha}'_{\mathfrak{p}}\right) \in \mathfrak{p}^{\mathbb{C}}.$$

As a consequence, F_U is the frame of a harmonic map $f_U : \tilde{M} \to U/(U \cap K^{\mathbb{C}})$, where actually

$$f_U = F_U \mod U \cap K^{\mathbb{C}}.$$

Computing the Birkhoff decomposition of F as well as of F_U we obtain

$$F_- F_+ = F = F_U W_+ = F_{U,-} \cdot F_{U,+} \cdot W_+,$$

with

$$F_- = I + O(\lambda^{-1}), \quad F_{U,-} = I + O(\lambda^{-1}) \in \Lambda_*^- U_\sigma^{\mathbb{C}} = \Lambda_*^- G_\sigma^{\mathbb{C}}.$$

This implies $F_- = F_{U,-}$, whence we also have $\eta = F_-^{-1} dF_- = F_{U,-}^{-1} dF_{U,-}$.

For the case of $\tilde{M} = S^2$, we use the extended frame constructed in the proof of Theorem 4.13. Then the argument can be repeated verbatim and we obtain a globally defined harmonic map $f_U : S^2 \to U/(U \cap K^{\mathbb{C}})$ and f shares the normalized potential with f_U. □

In view of the computations carried out above it is easy to verify that a harmonic map $f_U : M \to U/(U \cap K^{\mathbb{C}})$ is of finite uniton type if and only if its extended frame F actually is a Laurent polynomial in λ if and only if F_- is a Laurent polynomial in λ.

Corollary 4.23. *Let $f : \tilde{M} \to G/K$ be a harmonic map and f_U the associated harmonic map into the compact symmetric space $U/(U \cap K^{\mathbb{C}})$ as in Theorem 4.22. Then f is of finite uniton type if and only if f_U is of finite uniton type. Moreover, we have $r(f) = r(f_U)$.*

If $M = S^2$, then f always is of finite uniton type.

Proof. Let F, F_U be the (local) extended frame of f and f_U respectively as above. By (53), a Fourier expansion of F has only finite powers of λ^{-1} if and only if F_U has only finitely many powers of λ^{-1} with Fourier expansion. Using the reality of F and F_U, we conclude that F is a Laurent polynomial of λ if and only if F_U is a Laurent polynomial of λ. This is exactly the finite uniton number claim. The equality of $r(f)$ and $r(f_U)$ follows directly.

For the case of S^2, by Burstall and Guest's theory [12], f_U is of finite uniton type and hence also f is. □

Remark 4.24. Another way to show that f is of finite uniton type when $M = S^2$ is to follow Uhlenbeck's approach. Note that the proof of Theorem 11.5 of [57] can be applied verbatim when one assume G to be non-compact. Combining with the frame-free treatment in the proof of Theorem 4.13 in Section 4.2, this will give another proof of f being of finite uniton type when $M = S^2$.

5 Application of Loop group theory to Willmore surfaces

In this section we will present some examples which illustrate the theory of the previous sections.

5.1 Conformally harmonic maps containing a constant light-like vector

From Theorem 3.10, we see that there are two kinds of conformally harmonic maps satisfying $B_1^t I_{1,3} B_1 = 0$: those which contain a constant lightlike vector and those which do not contain a constant lightlike vector. Moreover if such a harmonic map f does not contain a lightlike vector, f will always be the conformal Gauss map of some Willmore map. This class of Willmore maps

corresponds exactly to all those Willmore maps which are not conformal to any minimal surface in $\mathbb{R}^{n+2}$, since minimal surfaces in $\mathbb{R}^{n+2}$ can be characterized as Willmore surfaces with their conformal Gauss map containing a constant lightlike vector. Since minimal surfaces in $\mathbb{R}^{n+2}$ can be constructed by various methods, we are mainly interested in Willmore surfaces not conformal to minimal surfaces in $\mathbb{R}^{n+2}$. It is therefore vital to derive a criterion to determine whether a strongly conformally harmonic map f contains a lightlike vector or not. This is the main goal of this subsection.

To begin with, we first prove a technical lemma.

Lemma 5.1. *Let*

$$A_1 = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & -a_{13} & -a_{14} \\ a_{13} & a_{13} & 0 & a_{34} \\ a_{14} & a_{14} & -a_{34} & 0 \end{pmatrix}$$

be a holomorphic matrix function on a contractible open Riemann surface U. Let F_{01} be a solution to the equation

$$F_{01}^{-1} dF_{01} = A_1 dz, \qquad F_{01}|_{z=0} = I_4.$$

Then

$$F_{01} = \begin{pmatrix} 1+\frac{1}{2}(b_{13}^2+b_{14}^2) & \frac{1}{2}(b_{13}^2+b_{14}^2) & b_{13} & b_{14} \\ -\frac{1}{2}(b_{13}^2+b_{14}^2) & 1-\frac{1}{2}(b_{13}^2+b_{14}^2) & -b_{13} & -b_{14} \\ b_{13} & b_{13} & 1 & 0 \\ b_{14} & b_{14} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\varphi & \sin\varphi \\ & & -\sin\varphi & \cos\varphi \end{pmatrix}$$

with

$$\varphi = \int_0^z a_{34}\ dw,$$

and

$$b_{13} = \int_0^z (a_{13}\cos\varphi + a_{14}\sin\varphi) dz, \quad \text{and} \quad b_{14} = \int_0^z (-a_{13}\sin\varphi + a_{14}\cos\varphi) dz.$$

Proof. Set

$$\tilde{F}_{01} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\varphi & \sin\varphi \\ & & -\sin\varphi & \cos\varphi \end{pmatrix} \text{ and } \hat{F}_{01} = \begin{pmatrix} 1+\frac{1}{2}(b_{13}^2+b_{14}^2) & \frac{1}{2}(b_{13}^2+b_{14}^2) & b_{13} & b_{14} \\ -\frac{1}{2}(b_{13}^2+b_{14}^2) & 1-\frac{1}{2}(b_{13}^2+b_{14}^2) & -b_{13} & -b_{14} \\ b_{13} & b_{13} & 1 & 0 \\ b_{14} & b_{14} & 0 & 1 \end{pmatrix}.$$

Straightforward computations yield

$$\tilde{F}_{01}^{-1} d\tilde{F}_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & -a_{34} & 0 \end{pmatrix} dz,$$

and

$$\hat{F}_{01}^{-1} d\hat{F}_{01} = \begin{pmatrix} 0 & 0 & b'_{13} & b'_{14} \\ 0 & 0 & -b'_{13} & -b'_{14} \\ b'_{13} & b'_{13} & 0 & 0 \\ b'_{14} & b'_{14} & 0 & 0 \end{pmatrix} dz,$$

with $b'_{13} = a_{13}\cos\varphi + a_{14}\sin\varphi$, $b'_{14} = -a_{13}\sin\varphi + a_{14}\cos\varphi$. Moreover, one obtains

$$\tilde{F}_{01}^{-1}\hat{F}_{01}^{-1}\hat{F}_{01}\tilde{F}_{01} = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & -a_{13} & -a_{14} \\ a_{13} & a_{13} & 0 & 0 \\ a_{14} & a_{14} & 0 & 0 \end{pmatrix}.$$

Since $F_{01}=\hat{F}_{01}\tilde{F}_{01}$, one derives

$$F_{01}^{-1}F_{01z}=\tilde{F}_{01}^{-1}\hat{F}_{01}^{-1}\hat{F}_{01}\tilde{F}_{01}+\tilde{F}_{01}^{-1}\tilde{F}_{01z}=A_1.$$

□

Theorem 5.2. *Let $\tilde{M}$ denote the Riemann surface $S^2,\mathbb{C}$ or the unit disk of $\mathbb{C}$. Let $f:\mathbb{D}\to SO^+(1,n+3)/SO^+(1,3)\times SO(n)$ be a strongly conformally harmonic map which contains a constant light-like vector. Assume that $f(p)=I_{n+4}\cdot K$ w.r.t some base point $p\in\tilde{M}$ and z is a local coordinate with $z(p)=0$. Then the normalized potential of f with reference point p is of the form*

$$\eta=\lambda^{-1}\begin{pmatrix}0 & \hat{B}_1\\ -\hat{B}_1^tI_{1,3} & 0\end{pmatrix}dz,\quad where\quad \hat{B}_1=\begin{pmatrix}\hat{f}_{11} & \hat{f}_{12} & \cdots & \hat{f}_{1n}\\ -\hat{f}_{11} & -\hat{f}_{12} & \cdots & -\hat{f}_{1n}\\ \hat{f}_{31} & \hat{f}_{32} & \cdots & \hat{f}_{3n}\\ i\hat{f}_{31} & i\hat{f}_{32} & \cdots & i\hat{f}_{3n}\end{pmatrix}. \tag{54}$$

Here f_{ij} are meromorphic functions on $\tilde{M}$.

Proof. Let $F(z,\bar{z},\lambda)=(\phi_1,\phi_2,\phi_3,\phi_4,\psi_1,\cdots,\psi_n)$ be a frame of f with the initial condition $F|_{z=0}=I_{n+4}$. W.l.g, we may assume that

$$Y_0=\phi_1-\phi_2$$

is the constant lightlike vector contained in f. As a consequence, we derive

$$\phi_{1z}=\phi_{2z}=a_{13}\phi_3+a_{14}\phi_4+\sqrt{2}\sum_{j=1}^{n}\beta_j\psi_j.$$

That is

$$(\phi_{1z},\phi_{2z})^t=\begin{pmatrix}0 & 0 & a_{13} & a_{14} & \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n\\ 0 & 0 & a_{13} & a_{14} & \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n\end{pmatrix}\cdot F^t$$

Comparing with $F^{-1}F_z=\begin{pmatrix}A_1 & B_1\\ -B_1^tI_{1,3} & A_2\end{pmatrix}$, we obtain

$$A_1=\begin{pmatrix}0 & 0 & a_{13} & a_{14}\\ 0 & 0 & -a_{13} & -a_{14}\\ a_{13} & a_{13} & 0 & a_{34}\\ a_{14} & a_{14} & -a_{34} & 0\end{pmatrix},\quad B_1=\begin{pmatrix}\sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n\\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n\\ -k_1 & \cdots & -k_n\\ -\hat{k}_1 & \cdots & -\hat{k}_n\end{pmatrix}.$$

Since $B_1^tI_{1,3}B_1=0$, $\hat{k}_1=ik_1,\cdots,\hat{k}_n=ik_n$ or $\hat{k}_1=-ik_1,\cdots,\hat{k}_n=-ik_n$. Similar to the discussion in Lemma 3.8 of Section 3, without loss of generality, we assume that on $\tilde{M}$, $\hat{k}_1=ik_1,\cdots,\hat{k}_n=ik_n$.

For the computation of the normalized potential, we will apply Wu's formula stated in Section 4.3. Let $\delta_1=(\tilde{a}_{ij})$ denote the "holomorphic part" of A_1 with respect to the base point $z=0$, i.e., the part of the Taylor expansion of A_1 which is independent of $\bar{z}$. Let F_{01} be a solution to the equation

$$F_{01}^{-1}dF_{01}=\delta_1dz,\ \ F_{01}|_{z=0}=I_4.$$

By Lemma 5.1,

$$F_{01}=\begin{pmatrix}1+\frac{1}{2}(b_{13}^2+\hat{a}_{14}^2) & \frac{1}{2}(b_{13}^2+\hat{a}_{14}^2) & b_{13}\cos\varphi-b_{14}\sin\varphi & b_{13}\sin\varphi+b_{14}\cos\varphi\\ -\frac{1}{2}(b_{13}^2+b_{14}^2) & 1-\frac{1}{2}(b_{13}^2+b_{14}^2) & -b_{13}\cos\varphi+b_{14}\sin\varphi & -(b_{13}\sin\varphi+b_{14}\cos\varphi)\\ b_{13} & b_{13} & \cos\varphi & \sin\varphi\\ b_{14} & b_{14} & -\sin\varphi & \cos\varphi\end{pmatrix}$$

with

$$\varphi = \int_0^z \tilde{a}_{34}\ dz,\ b_{13} = \int_0^z (\tilde{a}_{13}\cos\varphi + \tilde{a}_{14}\sin\varphi)dz,\ b_{14} = \int_0^z (-\tilde{a}_{13}\sin\varphi + \tilde{a}_{14}\cos\varphi)dz.$$

Let δ_2 denote the "holomorphic part" of A_2, with respect to the base point $z = 0$, and let F_{02} be a solution to the equation $F_{02}^{-1}dF_{02} = \delta_2 dz,\ F_{02}|_{z=0} = I_n$.

Let $\tilde{B}_1$ denote the holomorphic part of B_1. By Wu's formula (Theorem 4.17), the normalized potential can be represented in the form

$$\begin{aligned}\eta &= \lambda^{-1}\begin{pmatrix} F_{01} & 0 \\ 0 & F_{02}\end{pmatrix}\begin{pmatrix} 0 & \tilde{B}_1 \\ -\tilde{B}_1^t I_{1,3} & 0\end{pmatrix}\begin{pmatrix} F_{01} & 0 \\ 0 & F_{02}\end{pmatrix}^{-1} dz \\ &= \lambda^{-1}\begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0\end{pmatrix} dz,\end{aligned} \tag{55}$$

with

$$\hat{B}_1 = F_{01}\tilde{B}_1 F_{02}^{-1} = F_{01}\cdot\begin{pmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \cdots & \tilde{f}_{1n} \\ -\tilde{f}_{11} & -\tilde{f}_{12} & \cdots & -\tilde{f}_{1n} \\ -\tilde{f}_{31} & -\tilde{f}_{32} & \cdots & -\tilde{f}_{3n} \\ -i\tilde{f}_{31} & -i\tilde{f}_{32} & \cdots & -i\tilde{f}_{3n}\end{pmatrix}\cdot F_{02}^{-1} = \begin{pmatrix} \hat{f}_{11} & \hat{f}_{12} & \cdots & \hat{f}_{1n} \\ -\hat{f}_{11} & -\hat{f}_{12} & \cdots & -\hat{f}_{1n} \\ -\hat{f}_{31} & -\hat{f}_{32} & \cdots & -\hat{f}_{3n} \\ -i\hat{f}_{31} & -i\hat{f}_{32} & \cdots & -i\hat{f}_{3n}\end{pmatrix}.$$

It is not difficult to verify that $\eta(\frac{\partial}{\partial z})$ takes values in a nilpotent Lie subalgebra of rank 2 (see e.g. [27] for details). $\Box$

We note that the converse of Theorem 5.2 also holds.

Theorem 5.3. *([27]) Let η be a normalized potential of the form* (54)*. Then $B_1^t I_{1,3} B_1 = 0$ and we obtain a strongly conformally harmonic map $f : \tilde{M} \to SO^+(1, n+3)/SO^+(1,3)\times SO(n)$. Moreover, f contains a constant light-like vector and is of finite uniton type.*

Remark 5.4. For the proof of Theorem 5.3, one needs long and technical computations. It is easy to verify that f is of finite uniton type. Moreover, f actually belongs to one of the simplest cases, called $S^1 - invariant$ (See [12], [18]). For such harmonic maps, by using a direct but lengthy computation, one can derive the harmonic map directly and then read off the needed information. For details of the proof of Theorem 5.3 see [27].

Corollary 5.5. *Let $f : \tilde{M} \to SO^+(1, n+3)/SO^+(1,3)\times SO(n)$ be a strongly conformally harmonic map with its normalized potential η of the form* (54) *and of maximal rank$(\hat{B}_1) = 2$. Then f can not be the conformal Gauss map of a Willmore surface. In particular, there exist conformally harmonic maps which are not related to any Willmore map.*

Remark 5.6. Using the loop group method it is easy to see that harmonic maps satisfying the assumptions of the corollary always exist.

5.2 The conformal Gauss map of isotropic Willmore surfaces in S^4

Another important class of Willmore surfaces is formed by the *totally isotropic* Willmore surfaces.

Definition 5.7. *([16], [7], [31]) Let $y : M \to S^{n+2}$ be a smooth immersion, z a local coordinate of M and Y a local lift. Retaining the notation of Section 2.1, we denote by D_z^j the $j-$th derivative of κ. Then y is called* totally isotropic *if*

$$\langle D_z^j\kappa, D_z^l\kappa\rangle = 0,\ for\ j,\ l = 0,\ 1, \cdots. \tag{56}$$

Note that totally isotropic surfaces only exist in even dimensional spheres S^{2m}. They can be described as projections of holomorphic or anti-holomorphic curves in the twistor bundle $\mathfrak{T}S^{2m}\rightarrow S^{2m}$, see [31] for details.

It is well-known that isotropic surfaces in S^4 are all Willmore surfaces (moreover S-Willmore surfaces, see [31], [47]). However, in general, totally isotropic surfaces in S^{2m} are not necessarily Willmore surfaces when $m>2$. Until now, to the best of the knowledge of these authors, there does not exist a good geometric criterion to determine when an totally isotropic surface will be a Willmore surface.

Concerning isotropic (Willmore) surfaces in S^4, we have that

Theorem 5.8. *Let $y: M\rightarrow S^4$ be an isotropic surface from a simply connected Riemann surface $\tilde{M}$, with its conformal Gauss map $f=Gr$ defined in Section 2.1. Then the normalized potential of Gr is of the form*

$$\eta=\lambda^{-1}\left(\begin{array}{cc} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{array}\right)dz,\ \ with\ \hat{B}_1=\left(\begin{array}{cc} \hat{f}_{11} & i\hat{f}_{11} \\ \hat{f}_{21} & i\hat{f}_{21} \\ \hat{f}_{31} & i\hat{f}_{31} \\ \hat{f}_{41} & i\hat{f}_{41} \end{array}\right),\ -\hat{f}_{11}^2+\hat{f}_{21}^2+\hat{f}_{31}^2+\hat{f}_{41}^2=0. \quad (57)$$

Moreover, Gr is of finite uniton type with uniton number $r(f)$ at most 2. In particular, f is S^1-invariant.

Conversely, let η be defined on $\tilde{M}$ of the form (57) *and let $f:\tilde{M}\rightarrow SO^+(1,5)/SO^+(1,3)\times SO(2)$ be the associated strongly conformally harmonic map. Then either f is the conformal Gauss map of an isotropic S-Willmore surface in S^4, or f takes values in $SO^+(1,3)/SO^+(1,1)\times SO(2)$ or in $SO(4)/SO(2)\times SO(2)$ and is not the conformal Gauss map of any Willmore immersion.*

Proof. Retaining the notation of Section 2.1 for y and Gr, the isotropy property of y shows that $\langle\kappa,\kappa\rangle=0$. Differentiating with $D_{\bar{z}}$, one obtains $\langle D_{\bar{z}}\kappa,\kappa\rangle=0$. Noticing that the normal bundle is a line bundle, we observe that $D_{\bar{z}}\kappa$ is parallel to κ. Without loss of generality, we can assume

$$\kappa=k_1\psi_1+ik_1\psi_2,\ \text{and}\ D_{\bar{z}}\kappa=\beta_1\psi_1+i\beta_1\psi_2,$$

with ψ_1, ψ_2 an othonormal basis of sections of $V^{\perp}$ in the sense of Section 2.1. Therefore the Maurer-Cartan form of $F(z,\bar{z},\lambda)$ w.r.t y is

$$F^{-1}F_z=\left(\begin{array}{cc} A_1 & B_1 \\ -B_1^t I_{1,3} & A_2 \end{array}\right),\ \ \text{with}\ \ B_1=\left(\begin{array}{cc} \sqrt{2}\beta_1 & i\sqrt{2}\beta_1 \\ -\sqrt{2}\beta_1 & -i\sqrt{2}\beta_1 \\ -k_1 & -ik_1 \\ -ik_1 & -ik_1 \end{array}\right).$$

To apply Wu's formula (Theorem 4.17) as in the proof of Theorem 5.2, let δ_1, δ_2 and $\tilde{B}_1$ denote the "holomorphic part" of A_1, A_2 and $B_1$with respect to the reference point $z=0$ respectively, i.e., the part of the Taylor expansion of A_1, A_2 and B_1 which are independent of $\bar{z}$. Let F_{01} and F_{02} be the solutions to the equations $F_{01}^{-1}dF_{01}=\delta_1 dz$, $F_{01}|_{z=0}=I_4$ and $F_{02}^{-1}dF_{02}=\delta_2 dz$, $F_{02}|_{z=0}=I_2$ respectively. By Wu's formula (Theorem 4.17), the normalized potential can be represented in the form

$$\eta=\lambda^{-1}\left(\begin{array}{cc} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{array}\right)dz, \qquad \text{with} \qquad \hat{B}_1=F_{01}\tilde{B}_1F_{02}^{-1}.$$

Noticing that here $\tilde{B}_1$ is of the form

$$\tilde{B}_1 = (v_1, iv_1)\,, \quad \text{with} \quad v_1 \in \mathbb{C}^4_1,$$

it is immediate to check that both F_{01} and F_{02} will not change this form. So

$$\hat{B}_1 = (\hat{v}_1, i\hat{v}_1)\,.$$

It is easy to verify that $\hat{B}_1^t I_{1,3} \hat{B}_1 = 0$ is equivalent with $\hat{v}_1^t I_{1,3} \hat{v}_1 = 0$. This is exactly what was stated in (57).

The last statement is a corollary of the fact that $\eta(\frac{\partial}{\partial z})$ in (57) takes values in a nilpotent Lie subalgebra of rank 2 (see [27]). $\square$

Remark 5.9. Isotropic surfaces in S^4 provide another type of strongly conformally harmonic maps of finite uniton number ≤ 2, which actually have an intersection with minimal surfaces in $\mathbb{R}^4$ (see e.g. the examples below). For more details, we refer to the classification theorems in [51].

We also note that in [7] there has been derived a Weierstrass type representation for isotropic minimal surfaces in S^4.

As a consequence of Theorem 5.2, Theorem 5.8, and the classification theorems in [31], [52], [51], [46], we obtain

Corollary 5.10. *The conformal Gauss map of a Willmore two-sphere in S^4 is of finite uniton type with $r(y) \leq 2$ and hence S^1-invariant.*

Proof. By the classification theorems in [31], [52], [51], a Willmore two-sphere in S^4 is either isotropic or is conformal to a minimal surface in $\mathbb{R}^4$. Applying Theorem 5.2 and Theorem 5.8, we obtain the corollary. $\square$

Corollary 5.11. *The conformal Gauss map of a Willmore torus in S^4 with non-trivial normal bundle is of finite uniton number at most 2 and hence S^1-invariant.*

Proof. The main result of [46] states that a Willmore torus in S^4 with non-trivial normal bundle is either isotropic or is conformal to a minimal surface in $\mathbb{R}^4$. In the first case, the claim follows by Theorem 5.8 and the second case the claim follows by Theorem 5.2 and Theorem 5.3. $\square$

Here are two Willmore surfaces which are both isotropic and conformal to some minimal surfaces in $\mathbb{R}^4$.

Example 5.12. Let

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \quad \text{with} \quad \hat{B}_1 = \frac{1}{2} \begin{pmatrix} -i & 1 \\ i & -1 \\ z & iz \\ iz & -z \end{pmatrix}.$$

The corresponding associated family of Willmore surfaces is

$$y_\lambda = \frac{1}{1 + \frac{1}{r^2} + \frac{r^2}{4}} \begin{pmatrix} 1 - \frac{1}{r^2} - \frac{r^2}{4} \\ \frac{i(z-\bar{z})}{r^2} \\ \frac{-(z+\bar{z})}{r^2} \\ -\frac{i(\lambda^{-1}z - \lambda\bar{z})}{2} \\ \frac{\lambda^{-1}z + \lambda\bar{z}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda+\lambda^{-1}}{2} & \frac{\lambda-\lambda^{-1}}{-2i} \\ 0 & 0 & 0 & \frac{\lambda-\lambda^{-1}}{2i} & \frac{\lambda+\lambda^{-1}}{2} \end{pmatrix} \cdot y_1. \tag{58}$$

Note that y_λ is an embedded Willmore sphere in S^4 with Willmore energy $W(y_\lambda)=4\pi$ (recall the definition (14)). It is straightforward to verify that y_λ is conformal to the minimal graph

$$x_\lambda=\left(\frac{i(z-\bar z)}{r^2},\frac{-(z+\bar z)}{r^2},\ -\frac{i(\lambda^{-1}z-\lambda\bar z)}{2},\frac{\lambda^{-1}z+\lambda\bar z}{2}\right)^t \tag{59}$$

in $\mathbb{R}^4$. Together with the fact that y_λ is totally isotropic, it belongs to the intersection of isotropic surfaces in S^4 and surfaces conformal to minimal surfaces in $\mathbb{R}^4$. We refer the interested reader to [51] for a more detailed discussion of such surfaces.

Example 5.13. Let

$$\eta=\lambda^{-1}\left(\begin{array}{cc}0 & \hat B_1\\ -\hat B_1^t I_{1,3} & 0\end{array}\right)dz,\ \ \text{with}\ \ \hat B_1=\frac{1}{2}\left(\begin{array}{cc}-iz & z\\ iz & -z\\ 1 & i\\ i & -1\end{array}\right).$$

The corresponding associated family of Willmore surfaces is

$$y_\lambda=\frac{1}{1+r^2+\frac{r^4}{4}}\left(\begin{array}{c}1-r^2-\frac{r^4}{4}\\ -i(z-\bar z)\\ -(z+\bar z)\\ i(\lambda^{-1}\frac{z^2}{2}-\lambda\frac{\bar z^2}{2})\\ -(\lambda^{-1}\frac{z^2}{2}+\lambda\frac{\bar z^2}{2})\end{array}\right)=\left(\begin{array}{ccccc}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&\frac{\lambda+\lambda^{-1}}{2}&\frac{\lambda-\lambda^{-1}}{-2i}\\0&0&0&\frac{\lambda-\lambda^{-1}}{2i}&\frac{\lambda+\lambda^{-1}}{2}\end{array}\right)\cdot y_1. \tag{60}$$

This surface is isotropic and conformal to the minimal surface

$$x_\lambda=\left(-i(z-\bar z),\ -(z+\bar z),\ i(\lambda^{-1}\frac{z^2}{2}-\lambda\frac{\bar z^2}{2}),\ -(\lambda^{-1}\frac{z^2}{2}+\lambda\frac{\bar z^2}{2})\right)^t \tag{61}$$

in $\mathbb{R}^4$. Hence it belongs to the intersection of isotropic surfaces in S^4 and of surfaces conformal to minimal surfaces in $\mathbb{R}^4$. Note that y_λ has a branch point at ∞ and the Willmore energy is $W(y_\lambda)=2\pi$.

5.3 A concrete, new, and totally isotropic Willmore sphere in S^6

Since the work of Ejiri [31], it has been an open question whether there are Willmore spheres in S^n, $n>4$, which are not S-Willmore. After some tedious calculations, applying [12] in the spirit of [26], we show that there exist two different kinds of new Willmore 2-spheres in S^6. The first class of surfaces contains totally isotropic non S-Willmore Willmore spheres which are full in S^6. The other type of Willmore 2-spheres is more complicated. These Willmore surfaces are neither S-Willmore, nor have they an isotropic Hopf differential. In general, these Willmore surfaces may have branch points. Moreover, the latter type of surfaces may exist as full surfaces in some $S^5\subset S^6$ (No concrete examples have been worked out so far). Till now we only can show the existence of such (possibly branched) surfaces.

For the first type of new Willmore spheres in S^6, we have the following example:

Theorem 5.14. *([27]) Let*

$$\eta=\lambda^{-1}\left(\begin{array}{cc}0 & \hat B_1\\ -\hat B_1^t I_{1,3} & 0\end{array}\right)dz,\quad \text{with}\quad \hat B_1=\frac{1}{2}\left(\begin{array}{cccc}2iz & -2z & -i & 1\\ -2iz & 2z & -i & 1\\ -2 & -2i & -z & -iz\\ 2i & -2 & -iz & z\end{array}\right).$$

Then the associated family of Willmore two-spheres x_λ, $\lambda \in S^1$, corresponding to η, is

$$
\begin{aligned}
x_\lambda &= \frac{1}{\left(1+r^2+\frac{5r^4}{4}+\frac{4r^6}{9}+\frac{r^8}{36}\right)}
\begin{pmatrix}
\left(1-r^2-\frac{3r^4}{4}+\frac{4r^6}{9}-\frac{r^8}{36}\right)\\
-i\left(z-\bar z)(1+\frac{r^6}{9})\right)\\
\left(z+\bar z)(1+\frac{r^6}{9})\right)\\
-i\left((\lambda^{-1}z^2-\lambda\bar z^2)(1-\frac{r^4}{12})\right)\\
\left((\lambda^{-1}z^2+\lambda\bar z^2)(1-\frac{r^4}{12})\right)\\
-i\frac{r^2}{2}(\lambda^{-1}z-\lambda\bar z)(1+\frac{4r^2}{3})\\
\frac{r^2}{2}(\lambda^{-1}z+\lambda\bar z)(1+\frac{4r^2}{3})
\end{pmatrix}\\
&= \begin{pmatrix}
1&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
0&0&1&0&0&0&0\\
0&0&0&\frac{\lambda+\lambda^{-1}}{2}&\frac{\lambda-\lambda^{-1}}{-2i}&0&0\\
0&0&0&\frac{\lambda-\lambda^{-1}}{2i}&\frac{\lambda+\lambda^{-1}}{2}&0&0\\
0&0&0&0&0&\frac{\lambda+\lambda^{-1}}{2}&\frac{\lambda-\lambda^{-1}}{-2i}\\
0&0&0&0&0&\frac{\lambda-\lambda^{-1}}{2i}&\frac{\lambda+\lambda^{-1}}{2}
\end{pmatrix}\cdot x_1\\
&= D_\lambda\cdot x_1,
\end{aligned}
\tag{62}
$$

with $r=|z|$, $x_1 = x_\lambda|_{\lambda=1}$, and $D_\lambda \in SO(7)$. Note that all these surfaces x_λ, $\lambda\in S^1$, are isometric to each other by rotations by matrices of $SO(7)$. Moreover, $x_\lambda : S^2 \to S^6$ is a Willmore immersion in S^6, which is non S-Willmore, full, and totally isotropic. In particular, x_λ does not have any branch points.

Remark 5.15. For the proof of Theorem 5.14 and more discussion on Willmore two spheres, we refer to [27]. Here we note that this is the first new example of a Willmore 2-sphere in S^6 which is not S-Willmore (and thus has no dual surfaces).

Although x_1 has no dual surfaces, it has infinitely many adjoint surfaces, according to the discussion in Section 4.2 of [47]. And we note that

$$
\hat x = \frac{1}{\left(-r^2-1-\frac{r^4}{4}-\frac{r^6}{9}\right)}
\begin{pmatrix}
\left(-r^2+1+\frac{r^4}{4}+\frac{r^6}{9}\right)\\
i\left(\frac{r^2}{2}(z-\bar z)\right)\\
-\left(\frac{r^2}{2}(z+\bar z)\right)\\
i\left(\frac{r^2}{3}(z^2-\bar z^2)\right)\\
-\left(\frac{r^2}{3}(z^2+\bar z^2)\right)\\
-i\left((z-\bar z)\right)\\
\left((z+\bar z)\right)
\end{pmatrix},
$$

is an adjoint surface of x_1 and it is branched at ∞. The details will be published elsewhere.

5.4 Willmore immersions with symmetries

Among all Willmore immersions into S^{n+2}, some have additional symmetries. In this subsection we will briefly outline the general theory and in the following section we will present one of the many applications.

Let $y : M \to S^{n+2}$ be a Willmore immersion from a Riemann surface M to S^{n+2}. Assume that R is an orientation preserving conformal transformation of S^{n+2}, considered as a linear

transformation of $\mathbb{R}_1^{n+4}$. Assume $R(y(M)) = y(M)$. Then R expresses some "symmetry" of y. If M is simply connected and the metric induced on M by y is complete, then [21] there exists some orientation preserving conformal transformation γ of M with respect to the induced metric such that

$$y(\gamma.p) = Ry(p) \tag{63}$$

for all $p \in M$.

Therefore, if we talk about a symmetry of some Willmore immersion we always mean a pair (γ, R) as above. Actually, it is not necessary here to assume that M is simply connected nor that the induced metric is complete.

Theorem 5.16. *Let $y : M \to S^{n+2}$ be a Willmore immersion and (γ, R) a symmetry of y. Then the conformal Gauss map f of y satisfies*

$$f(\gamma.p) = \hat{R}f(p) \tag{64}$$

where $\hat{R}$ denotes the isometry of the symmetric space $SO^+(1, n+3)/SO^+(1, 3) \times SO(n)$, which is induced by R considered as an element of the Lorentz group $SO^+(1, n+3)$.

Proof. This follows directly from the construction of the conformal Gauss map. $\square$

Symmetries have been investigated for many types of harmonic maps. In our case we obtain

Theorem 5.17. *Let $f : M \to SO^+(1, n+3)/SO^+(1, 3) \times SO(n)$ be a harmonic map with symmetry $\left(\gamma, \hat{R}\right)$. Let F denote the moving frame associated with f as in Section 3. Then there exists some $k : M \to SO^+(1, 3) \times SO(n)$ such that*

$$\gamma^* F(p) = \hat{R}F(p)k(p). \tag{65}$$

Proof. It follows directly from (64). $\square$

Now the symmetry equation for F above implies

$$\gamma^* \alpha = k^{-1}\alpha k + k^{-1}dk \tag{66}$$

Clearly, $\gamma^*\alpha_1' = k^{-1}\alpha_1' k$, $\gamma^*\alpha_1'' = k^{-1}\alpha_1'' k$, and $\gamma^*\alpha_0 = k^{-1}\alpha_0 k + k^{-1}dk$. Recalling now how the spectral parameter λ was introduced (21), it follows

$$\gamma^* \alpha_\lambda = k^{-1}\alpha_\lambda k + k^{-1}dk \tag{67}$$

From this we obtain (see also e.g. [21])

Theorem 5.18. *Let M be simply connected and $(\gamma, \hat{R})$ a symmetry of the harmonic map $f : M \to SO^+(1, n+3)/SO^+(1, 3) \times SO(n)$. Then there exists some $\chi(\lambda) \in (\Lambda SO^+(1, n+3)_\sigma)^0$ such that*

$$\gamma^* F(z, \bar{z}, \lambda) = \chi(\lambda)F(z, \bar{z}, \lambda)k(z, \bar{z}), \tag{68}$$

where k is as above and thus independent of λ.

Proof. Since M is simply connected, the system of partial differential equations $dF(z, \bar{z}, \lambda) = F(z, \bar{z}, \lambda)\alpha_\lambda$ has a solution on all of M. By (67) it is clear that $\gamma^*F(z, \bar{z}, \lambda)$ and $F(z, \bar{z}, \lambda)$ solve the same system of pde's. Hence there exists some χ such that (68) holds. The right factor of equation (68) is obviously the same as for the case $\lambda = 1$. Since we have normalized $F(z, \bar{z}, \lambda)$ to satisfy $F(p, \lambda) = I$, equation (68) implies $\chi(\lambda) = F(\gamma.p, \lambda)$. In particular, $\chi(\lambda) \in (\Lambda SO^+(1, n+3)_\sigma)^0$. $\square$

To determine how the normalized potential behaves under the action of symmetries we perform a Birkhoff decomposition of $F_\lambda = F_- F_+$ (away from some discrete subset) and obtain, by collecting all elements in $\Lambda^+ SO^+(1, n+3, \mathbb{C})_\sigma$ in one place:

Theorem 5.19. *Let $f : M \to SO^+(1, n+3)/SO^+(1,3) \times SO(n)$ be a harmonic map with symmetry (γ, R). Then for F_- we have*

$$\gamma^* F_- = \chi F_- V_+ \tag{69}$$

for χ as above and some $V_+ \in \Lambda^+ SO^+(1, n+3, \mathbb{C})_\sigma$. For the normalized potential η of f we obtain

$$\gamma^* \eta = V_+^{-1} \eta V_+ + V_+^{-1} dV_+. \tag{70}$$

Proof. First, from Theorem 5.18, we have

$$\gamma^* F_\lambda = \chi F_\lambda k.$$

Therefore

$$\gamma^* F_- = \chi F_\lambda k \tilde{V}_+ = \chi F_- F_+ k \tilde{V}_+ = \chi F_- V_+, \quad \text{with} \quad V_+ = F_+ k \tilde{V}_+ \in \Lambda^+ SO^+(1, n+3, \mathbb{C})_\sigma.$$

Hence

$$\gamma^* \eta = (\gamma^* F_-)^{-1} d(\gamma^* F_-) = V_+^{-1} \eta V_+ + V_+^{-1} dV_+.$$

□

In general this is a very complicated equation. When using a holomorphic potential in place of the normalized potential, the situation will, in general, be equally complicated. Hence it is important to know that, by some clever choice of the potential, one can assume $V_+ = I$, if the symmetry belongs to the fundamental group of some surface M. However, this is of no importance to this paper, since we are only interested in examples, for which we can choose potentials that work.

For our applications it will be important to have the following theorem, which comes directly from the proof of Theorem 5.19.

Theorem 5.20. *Assume η is a potential for some harmonic map $f : M \to SO^+(1, n+3)/SO^+(1,3) \times SO(n)$. Assume moreover that γ is a conformal automorphism of M and that equation* (70) *holds, then there exists some $\chi(\lambda) \in (\Lambda SO^+(1, n+3, \mathbb{C})_\sigma)^0$ such that for the solution F_- to*

$$dF_- = F_- \eta, \quad F_-(z = 0, \lambda) = I$$

equation (69) *holds.*

Proof. Since $\gamma^* F_-$ and $F_- V_+$ satisfy the same ODE by (69), they only differ by some matrix in $\Lambda SO^+(1, n+3, \mathbb{C})_\sigma$ which does not depend on z and $\bar{z}$. As in the proof of Theorem 5.19 this matrix $\chi(\lambda)$ is contained in the connected loop group. □

Since it is of particular importance we state the following result as a separate theorem

Theorem 5.21. *With the notation of the previous theorem, if $\chi(\lambda) \in (\Lambda SO^+(1, n+3)_\sigma)^0$, then γ induces the symmetry*

$$\gamma^* f = \chi(\lambda) f \tag{71}$$

of the harmonic map f.

Proof. From Theorem 5.20, we obtain

$$\gamma^* F = \gamma^* F_- \gamma^*(V_+^{-1}) = \chi(\lambda) F_- V_+ \gamma^*(V_+^{-1}) = \chi(\lambda) F W_+$$

where $W_+ = V_+ \gamma^*(V_+^{-1})$. But since $\gamma^* F$, $\chi(\lambda)$ and F are in the real loop group $(\Lambda SO^+(1, n+3)_\sigma)^0$, also W_+ is in the real loop group $(\Lambda SO^+(1, n+3)_\sigma)^0$, whence W_+ is independent of λ and thus contained in the stabilizer group $SO^+(1,3) \times SO(n)$ of the symmetric space $SO^+(1, n+3)/SO^+(1,3) \times SO(n)$. $\square$

Since we are primarily interested in Willmore immersions we finally state

Theorem 5.22. *With the notation of the previous theorem, if f induces a unique Willmore immersion $y = [Y]$ into S^{n+2} (i.e., y is not S-Willmore or the dual surface of y reduces to a point), then equation* (71) *induces the symmetry,*

$$\gamma^* y = [\gamma^* Y] = [\chi(\lambda) Y], \tag{72}$$

since $\chi(\lambda) \in (\Lambda SO^+(1, n+3)_\sigma)^0$. If f induces a pair of dual Willmore immersion $y = [Y]$ and $\hat{y} = [\hat{Y}]$ into S^{n+2}, then equation (71) *induces the symmetry*

$$\gamma^* y := [\gamma^* Y] = [\chi(\lambda) Y] \quad \textit{and} \quad \gamma^* \hat{y} := [\gamma^* \hat{Y}] = [\chi(\lambda) \hat{Y}], \tag{73}$$

or

$$\gamma^*(\gamma^* y) = [\chi(\lambda)^2 Y] \quad \textit{and} \quad \gamma^*(\gamma^* \hat{y}) := [\chi(\lambda)^2 \hat{Y}]. \tag{74}$$

Moreover, for the latter case, equation (71) *also induces a symmetry between y and $\hat{y}$*

$$\gamma^* y = [\chi(\lambda) \hat{Y}] \quad \textit{and} \quad \gamma^* \hat{y} := [\gamma^* \hat{Y}] = [\chi(\lambda) \hat{Y}]. \tag{75}$$

Proof. Let Y be a lift of y with f as its conformal Gauss map. Let $\hat{y} = [\hat{Y}]$ denote the (non-degenerate) dual surface of y if there exists. Let F be a local lift of f with its Maurer-Cartan form $\alpha = F^{-1} dF$ of the form in Proposition 2.2. Moreover, we obtain $\pi_0(F) = [Y] = y$ from (12). Now from the proof of Theorem 5.21, one derives that $\gamma^* F = \chi(\lambda) F k$ for some $k = k(z, \bar{z})$. As a consequence, $\gamma^* \tilde{F} = \gamma^* F k^{-1}$ has the Maurer-Cartan form

$$\gamma^* F k^{-1} d(\gamma^* F k^{-1}) = (\chi(\lambda) F)^{-1} d(\chi(\lambda) F) = F^{-1} dF = \alpha.$$

So $\gamma^* f$ is the conformal Gauss map of $\pi_0(\gamma^* \tilde{F}) = \pi_0(\chi(\lambda) F) = [\chi(\lambda) Y]$. On the other hand, $\gamma^* f$ is also the conformal Gauss map of $\gamma^* y$ and $\gamma^* \hat{y}$ if $\hat{y}$ exists as an immersion. Therefore we obtain

$$[\gamma^* Y] = [\chi(\lambda) Y] \quad \text{or} \quad [\gamma^* \hat{Y}] = [\chi(\lambda) Y].$$

When $[\gamma^* \hat{Y}] = [\chi(\lambda) Y]$, for the same reason we also have $[\gamma^* Y] = [\chi(\lambda) \hat{Y}]$. As a consequence,

$$[\gamma^*(\gamma^* Y)] = [\chi(\lambda)^2 Y], \quad \text{and} \quad [\gamma^*(\gamma^* \hat{Y})] = [\chi(\lambda)^2 \hat{Y}].$$

$\square$

5.5 Willmore surfaces admitting one-parameter groups of isometries

Equivariant surfaces, i.e. surfaces admitting one-parameter groups of isometries have been discussed for many surface classes [13], [32]. In the case of Willmore surfaces the variety of such surfaces seems to be much larger than for the usual surface classes in $\mathbb{R}^3$ or other three dimensional space forms.

Consider the holomorphic potential

$$\eta = \lambda^{-1}\eta_{-1} + \eta_0 + \lambda\eta_1, \tag{76}$$

where we assume that $\eta \in \Lambda\mathfrak{g}_\sigma$. Then the solution to

$$dC = C\eta \quad \text{with intial condition} \quad C(z=0,\lambda) = e \tag{77}$$

has the property

$$C(z+t,\lambda) = \chi_t(\lambda)C(z,\lambda). \tag{78}$$

Since we have assumed $\eta \in \Lambda\mathfrak{g}_\sigma$, clearly $\chi_t(\lambda) = \exp(t\eta) \in \Lambda G_\sigma$ for all $t \in \mathbb{R}$.

As a consequence, the results above apply and we obtain the one parameter group $\chi_t(\lambda)$ acting on the harmonic map associated with η. Now let us look at the Iwasawa decomposition $C(z,\lambda) = F(z,\bar z,\lambda)F_+(z,\bar z,\lambda)$. We then derive from above

$$F(z,\bar z,\lambda) = e^{z\eta}\hat V(y,\lambda).$$

Putting

$$\alpha = F^{-1}dF = \alpha' dz + \alpha'' d\bar z,$$

this equation is equivalent with

$$\alpha' + \alpha'' = \hat V^{-1}\eta\hat V \tag{79}$$

$$i(\alpha' - \alpha'') = \hat V^{-1}i\eta\hat V + \hat V^{-1}d\hat V. \tag{80}$$

Inserting (79) into (80), we obtain

$$\hat V' = \hat V(i(\alpha' - \alpha'')) + i(\alpha' + \alpha'')\hat V. \tag{81}$$

More discussion on these three equations will yield detailed properties of equivariant Willmore surfaces, which we will leave for a future discussion. In this paper we only give some examples.

Example 5.23. Example 5.12 in (58), Example 5.13 in (60), and the new Willmore 2-sphere in (62), all admit a one-parameter group of isometries. To be concrete, let $\gamma_t(z) = ze^{it}$ for $z \in \mathbb{C}^*$. This provides a one-parameter subgroup of automorphisms of S^2, namely, the group of rotations on S^2 preserving 0 and ∞.

1. One can easily verify for Example 5.12

$$x_\lambda(\gamma_t(z)) = \hat R_t \cdot x_\lambda(z)$$

with

$$\hat R_t = \begin{pmatrix} 1 & & & & \\ & \cos t & -\sin t & & \\ & \sin t & \cos t & & \\ & & & \cos t & -\sin t \\ & & & \sin t & \cos t \end{pmatrix}.$$

2. For Example 5.13, we have

$$x_\lambda(\gamma_t(z)) = \hat R_t \cdot x_\lambda(z)$$

with

$$\hat{R}_t = \begin{pmatrix} 1 & & & & \\ & \cos t & -\sin t & & \\ & \sin t & \cos t & & \\ & & & \cos 2t & -\sin 2t \\ & & & \sin 2t & \cos 2t \end{pmatrix}.$$

2. For the example in (62), we have

$$x_\lambda(\gamma_t(z)) = \hat{R}_t \cdot x_\lambda(z)$$

with

$$\hat{R}_t = \begin{pmatrix} 1 & & & & & & \\ & \cos t & -\sin t & & & & \\ & \sin t & \cos t & & & & \\ & & & \cos 2t & -\sin 2t & & \\ & & & \sin 2t & \cos 2t & & \\ & & & & & \cos t & -\sin t \\ & & & & & \sin t & \cos t \end{pmatrix}.$$

Remark 5.24. The examples above show that it is possible to compute at least some examples quite explicitly. The general picture has many parameters and needs more computations. One of the very interesting cases of the above discussion are Willmore tori. This is another topic to be investigated separately.

5.6 Homogeneous examples

In the previous subsection we have given some examples for Willmore surfaces in S^{n+2}, which admit a one-parameter group of extrinsic isometries. Since in S^{n+2} the corresponding group $SO^+(1, n+3)$ of conformal transformations does contain abelian subgroups of dimension at least 2 for any choice of $n \in \mathbb{Z}^+$, one can also consider the case of Willmore surfaces which are invariant under a two-parameter abelian group.

Definition 5.25. *A Willmore immersion $y : \mathbb{D} \to S^{n+2}$ is called homogeneous if there exists a group $\Gamma := \{(\gamma, R_\gamma) : \ \gamma \in Aut\mathbb{D}, R \in SO^+(1, n+3)\}$ such that*

$$y(\gamma \cdot z) = R_\gamma \cdot y(z), \ \textit{for all } z \in \mathbb{D} \textit{ and } \ (\gamma, R_\gamma) \in \Gamma. \tag{82}$$

A general discussion of such surfaces can not be included into this paper. But we will construct some examples.

Let $\eta = (\lambda^{-1}B + A)dz$ with A, B constant matrices satisfying

$$[\lambda^{-1}B + A, \lambda\bar{B} + \bar{A}] = 0.$$

Then

$$e^{z(\lambda^{-1}B+A)} = e^{z(\lambda^{-1}B+A)+\bar{z}(\lambda\bar{B}+\bar{A})} \cdot e^{-\bar{z}(\lambda\bar{B}+\bar{A})}$$

is an Iwasawa decomposition, producing the extended frame

$$F(z, \bar{z}, \lambda) = e^{z(\lambda^{-1}B+A)+\bar{z}(\lambda\bar{B}+\bar{A})}.$$

Clearly,

$$\alpha = F^{-1}dF = (\lambda^{-1}B + A)dz + (\lambda\bar{B} + \bar{A})d\bar{z}.$$

Conversely, assume the Maurer-Cartan form α of some extended frame associated with some Willmore immersion has constant coefficient matrices. Write $\alpha = (\lambda^{-1}B + A)dz + (\lambda\bar{B} + \bar{A})d\bar{z}$, then the integrability condition for α implies that $[(\lambda^{-1}B + A), (\lambda\bar{B} + \bar{A})] = 0$, and $F(z, \bar{z}, \lambda) = e^{z(\lambda^{-1}B+A)+\bar{z}(\lambda\bar{B}+\bar{A})}$ with $F(0, 0, \lambda) = I_{n+4}$, follows.

It is easy to see that F is generated by the holomorphic potential $\eta = (\lambda^{-1}B + A)dz$, where $[(\lambda^{-1}B + A), (\lambda\bar{B} + \bar{A})] = 0$. In this case we obtain for all $z \in \mathbb{D}$, $u \in \mathbb{C}$:

$$y(z + u, \bar{z} + \bar{u}) = e^{z(\lambda^{-1}B+A)+\bar{z}(\lambda\bar{B}+\bar{A})} y(z, \bar{z}).$$

Recall that a vacuum solution [14]

$$\eta = (\lambda^{-1}B)dz, \quad \text{with } [B, \bar{B}] = 0,$$

produces a harmonic map f. For f being a strongly conformally harmonic map, one needs to assume that

$$B = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,3} & 0 \end{pmatrix}, \quad \text{with } \ B_1^t I_{1,3} B_1 = 0.$$

Moreover, as shown in Lemma 3.4, there exists some $L_1 \in SO^+(1, 3)$ such that $L_1 B_1$ is of the form stated in Lemma 3.4. By Theorem 3.10, one sees that the maximal rank of B_1 must be one if f is the conformal Gauss map of some Willmore map y. Hence we may assume that

$$B_1 = (\mathrm{v}_1, \cdots, \mathrm{v}_n) \ \text{ with } \ \mathrm{v}_j = (a_j + ib_j)\mathrm{v}_0, a_j, b_j \in \mathbb{R}, j = 1, \cdots, n.$$

Here $\mathrm{v}_0 \in \mathrm{Span}_{\mathbb{C}}\{(1, -1, 0, 0)^t, (0, 0, 1, i)^t\}$. Since the rank of $\mathrm{Span}_{\mathbb{R}}\{(a_1, b_1), \cdots, (a_n, b_n)\}$ is no more than 2, there exists some $L_2 \in SO(n)$ such that

$$B_1 L_2 = \mathrm{v}_0 \left(a_1 + ib_1, \cdots, a_n + ib_n\right) L_2 = \Big((\hat{a}_1 + i\hat{b}_1)\mathrm{v}_0, (\hat{a}_2 + i\hat{b}_2)\mathrm{v}_0, 0, \cdots, 0\Big).$$

As a consequence, we obtain

Proposition 5.26. *Let f be the conformal Gauss map of a Willmore surface. If f is a vacuum solution, then a normalized potential of f is*

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz \ \ \textit{with} \ \ \hat{B}_1 = \begin{pmatrix} c_1 & c_0 c_1 \\ -c_1 & -c_0 c_1 \\ c_3 & c_0 c_3 \\ ic_3 & ic_0 c_3 \end{pmatrix} \ \ \textit{and} \ \ c_0, \ c_1, \ c_3 \in \mathbb{C}. \tag{83}$$

One the other hand, when $A \neq 0$, the normalized potential associated with $\eta = (\lambda^{-1}B + A)dz$ is no longer constant, whence the corresponding harmonic map is not a vacuum solution. Therefore, potentials of the type $\eta = (\lambda^{-1}A + B)dz$, with $[(\lambda^{-1}A + B), (\lambda\bar{A} + \bar{B})] = 0$ and $B \neq 0$, produce homogeneous Willmore surfaces which cannot be generated by a vacuum. For Willmore surfaces of this type, we have the following examples, some of which appear for the first time to the authors' knowledge:

Example 5.27. Let $y = [Y] : S^1 \times R^1 \to S^4$ be a cylinder

$$Y = (\cosh av, \sinh av, \cos u \cos bv, \cos u \sin bv, \sin u \cos bv, \sin u \sin bv)^t \tag{84}$$

with $a^2 + b^2 = 1,\ a, b \in \mathbb{R}$. Note that when $a = 0$ we obtain the Clifford torus in $S^3 \subset S^4$, and when $b = 0$ we obtain the round sphere with the north pole removed. (For a detailed discussion on Willmore tori in S^4, we refer to [5] and [46].)

Direct computation shows $Y_z = \frac{1}{2}(e_1 - ie_2)$ with

$$\begin{cases} e_1 = (0, 0, -\sin u \cos bv, -\sin u \sin bv, \cos u \cos bv, \cos u \sin bv), \\ e_2 = (a \sinh av, a \cosh av, -b \cos u \sin bv, b \cos u \cos bv, -b \sin u \sin bv, b \sin u \cos bv). \end{cases}$$

And

$$\begin{cases} Y_{z\bar{z}} = \dfrac{-1-b^2}{4}\left(\dfrac{-a^2}{1+b^2}\cosh av, \dfrac{-a^2}{1+b^2}\sinh av, \cos u \cos bv, \cos u \sin bv, \sin u \cos bv, \sin u \sin bv\right), \\ Y_{zz} = \dfrac{-a^2}{4}(\cosh av, \sinh av, \cos u \cos bv, \cos u \sin bv, \sin u \cos bv, \sin u \sin bv) \\ \qquad - \dfrac{ib}{2}(0, 0, \sin u \sin bv, -\sin u \cos bv, -\cos u \sin bv, \cos u \cos bv). \end{cases}$$

So we see that $s = \frac{a^2}{2}$, $\kappa = -\frac{ib}{2}\psi_2$, with

$$\begin{cases} \psi_1 = (-b \sinh av, -b \cosh av, -a \cos u \sin bv, a \cos u \cos bv, -a \sin u \sin bv, a \sin u \cos bv). \\ \psi_2 = (0, 0, \sin u \sin bv, -\sin u \cos bv, -\cos u \sin bv, \cos u \cos bv). \end{cases}$$

It is straightforward to derive $D_z\psi_1 = \frac{a}{2}\psi_2$, $D_z\psi_2 = -\frac{a}{2}\psi_1$. So we verify easily that

$$D_{\bar{z}}D_{\bar{z}}\kappa = -\frac{a^2}{4}\kappa = -\frac{\bar{s}}{2}\kappa,$$

i.e., y is a Willmore immersion. Moreover, the Maurer-Cartan form is a constant matrix, therefore providing a holomorphic potential of y of the form

$$\tilde{\eta} = \frac{dz}{4\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & -i(1+2a^2) & 2iab\lambda^{-1} & 0 \\ 0 & 0 & 3 & -i(1+2b^2) & -2iab\lambda^{-1} & 0 \\ 1 & -3 & 0 & 0 & 0 & i2\sqrt{2}b\lambda^{-1} \\ -i(1+2a^2) & i(1+2b^2) & 0 & 0 & 0 & 2\sqrt{2}b\lambda^{-1} \\ 2iab\lambda^{-1} & 2iab\lambda^{-1} & 0 & 0 & 0 & -2\sqrt{2}a \\ 0 & 0 & -i2\sqrt{2}b\lambda^{-1} & -2\sqrt{2}b\lambda^{-1} & 2\sqrt{2}a & 0 \end{pmatrix}. \tag{85}$$

The following example of the Veronese sphere shows that not all homogeneous Willmore surfaces have an abelian transitive group.

Example 5.28. It is well-known that the Veronese sphere in S^4 is a homogeneous minimal surface. It is given by

$$y = \left(\frac{a^2+b^2-2c^2}{2}, \sqrt{3}ac, \sqrt{3}bc, \frac{\sqrt{3}(a^2-b^2)}{2}, \sqrt{3}ab\right)^t \tag{86}$$

with $a^2+b^2+c^2 = 1$. Setting $a = \frac{z+\bar{z}}{1+r^2}$, $b = \frac{-i(z+\bar{z})}{1+r^2}$, $c = \frac{1-r^2}{1+r^2}$, with $r = |z|$, $z \in \mathbb{C}$, we can re-parameterize y in terms of the conformal coordinate z on $\mathbb{C} \subset S^2$, that is

$$y = \left(\frac{2r^2-(1-r^2)^2}{(1+r^2)^2}, \frac{\sqrt{3}(z+\bar{z})(1-r^2)}{(1+r^2)^2}, \frac{-i\sqrt{3}(z-\bar{z})(1-r^2)}{(1+r^2)^2}, \frac{\sqrt{3}(z^2+\bar{z}^2)}{(1+r^2)^2}, \frac{-i\sqrt{3}(z^2-\bar{z}^2)}{(1+r^2)^2}\right). \tag{87}$$

Computing the structure equations of y in S^4 yields $|y_z|^2 = \frac{1}{2}e^{2\omega} = \frac{6}{(1+r^2)^2}$, and

$$\begin{cases} y_{zz} &= 2\omega_z y_z + \Omega E, \\ y_{z\bar{z}} &= -\frac{1}{2}e^{2\omega}y, \\ E_z &= \alpha E, \\ \bar{E}_z &= -\alpha\bar{E} - 4e^{-2\omega}\Omega y_{\bar{z}}. \end{cases}$$

with $\{y, y_z, y_{\bar{z}}, E, \bar{E}\}$ providing a moving frame of y in S^4. And the integrability equations are

$$-K+1 = 8e^{-4\omega}|\Omega|^2, \quad \Omega_{\bar{z}} = \bar{\alpha}\Omega, \quad \alpha_{\bar{z}} + \bar{\alpha}_z = -4|\Omega|^2e^{-2\omega}.$$

It is straightforward to derive

$$\omega = \frac{1}{2}(\ln 12 - 2\ln(1+r^2)),\ \omega_z = -\frac{\bar{z}}{1+r^2},\ \omega_{zz} = \frac{\bar{z}^2}{(1+r^2)^2},\ \omega_{z\bar{z}} = -\frac{1}{(1+r^2)^2}.$$

$$K = -4e^{-2\omega}\omega_{z\bar{z}} = \frac{1}{3} \ \Rightarrow\ \Omega^2 = \frac{1}{12}e^{4\omega} = \frac{12}{(1+r^2)^4},\ \Omega = \frac{2\sqrt{3}}{(1+r^2)^2},\ \alpha = \frac{\Omega_z}{\Omega} = \frac{-2\bar{z}}{1+r^2}.$$

Setting $Y = e^{-\omega}(1, y)$, one obtains directly

$$s = 2\omega_{zz} - 2\omega_z^2 = 0, \quad \kappa = e^{-\omega}\Omega(\psi_1 - i\psi_2) = \frac{1}{1+r^2}(\psi_1 - i\psi_2),$$

and

$$D_{\bar{z}}\kappa = \frac{-z}{(1+r^2)^2}(\psi_1 - i\psi_2) + (-\bar{\alpha})\frac{1}{1+r^2}(\psi_1 - i\psi_2) = \frac{1}{(1+r^2)^2}(\psi_1 - i\psi_2), \quad D_{\bar{z}}D_{\bar{z}}\kappa = 0.$$

Introducing loop $\lambda \in S^1$, we have $Y_\lambda =$

$$\left(1+r^2, \frac{2r^2-(1-r^2)^2}{1+r^2}, \frac{\sqrt{3}(z+\bar{z})(1-r^2)}{1+r^2}, \frac{-i\sqrt{3}(z-\bar{z})(1-r^2)}{1+r^2}, \frac{\sqrt{3}(\lambda^{-1}z^2+\lambda\bar{z}^2)}{1+r^2}, \frac{-i\sqrt{3}(\lambda^{-1}z^2-\lambda\bar{z}^2)}{1+r^2}\right).$$

By above computations it is straightforward to verify that $F_\lambda^{-1}F_{\lambda z} =$

$$\begin{pmatrix} 0 & 0 & \frac{1}{2\sqrt{2}}(1-\frac{4}{(1+r^2)^2}) & \frac{-i}{2\sqrt{2}}(1-\frac{4}{(1+r^2)^2}) & \frac{\sqrt{2}\lambda^{-1}z}{(1+r^2)^2} & \frac{-\sqrt{2}i\lambda^{-1}z}{(1+r^2)^2} \\ 0 & 0 & \frac{1}{2\sqrt{2}}(1+\frac{4}{(1+r^2)^2}) & \frac{-i}{2\sqrt{2}}(1+\frac{4}{(1+r^2)^2}) & \frac{-\sqrt{2}\lambda^{-1}z}{(1+r^2)^2} & \frac{\sqrt{2}i\lambda^{-1}z}{(1+r^2)^2} \\ \frac{1}{2\sqrt{2}}(1-\frac{4}{(1+r^2)^2}) & \frac{-1}{2\sqrt{2}}(1+\frac{4}{(1+r^2)^2}) & 0 & 0 & \frac{-\lambda^{-1}}{1+r^2} & \frac{i\lambda^{-1}}{1+r^2} \\ \frac{-i}{2\sqrt{2}}(1-\frac{4}{(1+r^2)^2}) & \frac{i}{2\sqrt{2}}(1+\frac{4}{(1+r^2)^2}) & 0 & 0 & \frac{-i\lambda^{-1}}{1+r^2} & \frac{-\lambda^{-1}}{1+r^2} \\ \frac{\sqrt{2}\lambda^{-1}z}{(1+r^2)^2} & \frac{\sqrt{2}\lambda^{-1}z}{(1+r^2)^2} & \frac{\lambda^{-1}}{1+r^2} & \frac{i\lambda^{-1}}{1+r^2} & 0 & \frac{-2i\bar{z}}{1+r^2} \\ \frac{-\sqrt{2}i\lambda^{-1}z}{(1+r^2)^2} & \frac{-\sqrt{2}i\lambda^{-1}z}{(1+r^2)^2} & \frac{-i\lambda^{-1}}{1+r^2} & \frac{\lambda^{-1}}{1+r^2} & \frac{2i\bar{z}}{1+r^2} & 0 \end{pmatrix},$$

with $F_\lambda = \left(\frac{Y_\lambda+N_\lambda}{\sqrt{2}}, \frac{-Y_\lambda+N_\lambda}{\sqrt{2}}, Y_{\lambda z}+Y_{\lambda\bar{z}}, -i(Y_{\lambda z}-Y_{\lambda\bar{z}}), \psi_{\lambda 1}, \psi_{\lambda 2}\right)$ and initial condition $F_\lambda|_{z=0} = I$. Therefore we derive

Proposition 5.29. *The normalized potential $\eta = \lambda^{-1}\eta_{-1}dz$ of y relative to the base point $z=0$ is*

$$\eta_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{5z}{\sqrt{2}} & \frac{-5iz}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{-7z}{\sqrt{2}} & \frac{7iz}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -1+3z^2 & i-3iz^2 \\ 0 & 0 & 0 & 0 & -i-3iz^2 & -1-3z^2 \\ \frac{5z}{\sqrt{2}} & \frac{7z}{\sqrt{2}} & 1-3z^2 & i-3iz^2 & 0 & 0 \\ \frac{-5iz}{\sqrt{2}} & \frac{-7iz}{\sqrt{2}} & -i-3iz^2 & 1+3iz^2 & 0 & 0 \end{pmatrix}. \tag{88}$$

Proof. By Wu's formula, we have

$$\delta_0 = \frac{\sqrt{2}}{4}\begin{pmatrix} 0 & 0 & -3 & 3i & 0 & 0 \\ 0 & 0 & 5 & -5i & 0 & 0 \\ -3 & -5 & 0 & 0 & 0 & 0 \\ 3i & 5i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \delta_1 = \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz,\ \hat{B}_1 = \begin{pmatrix} \sqrt{2}z & -\sqrt{2}iz \\ -\sqrt{2}z & \sqrt{2}iz \\ -1 & i \\ -i & -1 \end{pmatrix}$$

Then we have the solution to $F_0^{-1}dF_0 = \delta_0 dz$, $F_0|_{z=0} = I$, is

$$F_0 = \begin{pmatrix} 1 & 0 & -\frac{3z}{2\sqrt{2}} & \frac{3iz}{2\sqrt{2}} & 0 & 0 \\ 0 & 1 & \frac{5z}{2\sqrt{2}} & \frac{-5iz}{2\sqrt{2}} & 0 & 0 \\ \frac{-3z}{2\sqrt{2}} & \frac{-5z}{2\sqrt{2}} & 1-z^2 & iz^2 & 0 & 0 \\ \frac{3iz}{2\sqrt{2}} & \frac{5iz}{2\sqrt{2}} & iz^2 & 1+z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Substituting into Wu's formula, we obtain $\eta_{-1} = F_0\delta_1 F_0^{-1}$, finishing the proof. □

6 Appendix A: Two decomposition theorems

In this section we list the relevant facts about the decomposition theorems of the loop group $\Lambda SO^+(1,\tilde{n},\mathbb{C})_\sigma$ with $\tilde{n} = n+3$. We will always assume $4 \le 1+\tilde{n} = 2m$. In particular, $\tilde{n}$ is an odd integer and $3 \le \tilde{n}$. One of the main result is that there are exactly two open Iwasawa "big cells" in the loop group $\Lambda SO^+(1,\tilde{n},\mathbb{C})_\sigma$.

6.1 Birkhoff Decomposition

For the loop group method used in this paper two decomposition theorems are of crucial importance. The first is

Theorem 6.1. *(Birkhoff decomposition theorem) Let $G^{\mathbb{C}}$ denote a simply connected complex Lie group with connected real form G and let σ be an inner involution of G and $G^{\mathbb{C}}$. Then the multiplication*

$$\Lambda_*^- G_\sigma^{\mathbb{C}} \times \Lambda^+ G_\sigma^{\mathbb{C}} \to \Lambda_*^- G_\sigma^{\mathbb{C}} \cdot \Lambda^+ G_\sigma^{\mathbb{C}} \tag{89}$$

is a complex analytic diffeomorphism and the (left) "big cell" $\Lambda_*^- G_\sigma^{\mathbb{C}} \cdot \Lambda^+ G_\sigma^{\mathbb{C}}$ is open and dense in $\Lambda G_\sigma^{\mathbb{C}}$. More precisely, every g in $\Lambda G_\sigma^{\mathbb{C}}$ can be written in the form

$$g = g_- \cdot \omega \cdot g_+ \tag{90}$$

with $g_\pm \in \Lambda^\pm G_\sigma^{\mathbb{C}}$, and $\omega \in \mathfrak{W}$, the Weyl group of $\Lambda G_\sigma^{\mathbb{C}}$.

Proof. The decomposition above has been proven for algebraic loop groups in in [41]. Our results follow by completeness in the Wiener Topology (see e.g. [26]). □

Remark 6.2. Let J denote a nondegenerate quadratic form in $\mathbb{R}^{2m}$ and $SO(J)$ the corresponding real special orthogonal group. Let $SO(J,\mathbb{C})$ denote the complexified special orthogonal group. Then $SO(J,\mathbb{C})$ is connected and has fundamental group $\pi_1(SO(J,\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$. Moreover, since σ is an inner automorphism, we have $\Lambda SO(J,\mathbb{C}) \cong \Lambda SO(J,\mathbb{C})_\sigma$. Therefore

$$\pi_0(\Lambda SO(J,\mathbb{C})_\sigma) \cong \pi_0(\Lambda SO(J,\mathbb{C})) \cong \pi_1(SO(J,\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}. \tag{91}$$

The loop group $\Lambda SO^+(1,\tilde{n},\mathbb{C})_\sigma$ thus has two connected components.

Consider the simply connected cover $\pi : Spin(2m, \mathbb{C}) \to SO(2m, \mathbb{C})$. Then π induces a homomorphism from $\Lambda Spin(2m, \mathbb{C})_\sigma$ to $(\Lambda SO(2m, \mathbb{C})_\sigma)^0$, the identity component of $\Lambda SO(2m, \mathbb{C})_\sigma$. We will simply write $\Lambda SO(2m, \mathbb{C})^0_\sigma$ for $(\Lambda SO(2m, \mathbb{C})_\sigma)^0$. Applying Theorem 6.1 for $\Lambda SO(2m, \mathbb{C})^0_\sigma$, we obtain

Theorem 6.3. *If one replaces in the theorem above the group $G^{\mathbb{C}}$ by $\Lambda SO(2m, \mathbb{C})^0_\sigma$, then the statements of the Birkhoff Decomposition Theorem still hold.*

Remark 6.4. 1. If we represent $SO^+(1, \tilde{n}, \mathbb{C})$ as the special orthogonal group $SO(J)$ relative to the quadratic form $J = \text{off-diag}\{1, \cdots, 1\}$ (see [27]), then it is quite easy to give an explicit description of $\mathfrak{W}$ in $\Lambda SO(J)^{\mathbb{C}}_\sigma$. We will not need such an explicit description of $\mathfrak{W}$ in this paper.

2. Note that the Birkhoff factorization induced from $\Lambda Spin(2m, \mathbb{C})_\sigma$ only covers the identity component of $\Lambda SO^+(1, \tilde{n}, \mathbb{C})_\sigma$. But this is exactly what we need in our geometric applications, since we consider only connected surfaces and assume $F(z_0, \lambda) = I$ for some base point z_0.

3. Much of the above is contained in [55], Section 8.5 (see also [56]). Note, however, that our real group $G = SO^+(1, \tilde{n})$ is not compact.

6.2 Iwasawa Decomposition

For our geometric applications we also need a second loop group decomposition. Ideally we would like to be able to write any $g \in \Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$ in the form $g = hv_+$ with $h \in \Lambda SO^+(1, \tilde{n})^0_\sigma$ and $v_+ \in \Lambda^+ SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$. Unfortunately, this is not always possible. We thus obtain

$$\Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma = \bigcup_{\delta \in \Xi} \Lambda SO^+(1, \tilde{n})^0_\sigma \cdot \delta \cdot \Lambda^+ SO^+(1, \tilde{n}, \mathbb{C})_\sigma, \tag{92}$$

where Ξ is a complete set of representatives of the orbits (called Iwasawa cells) under the group action of $\Lambda SO^+(1, \tilde{n})^0_\sigma \times \Lambda^+ SO^+(1, \tilde{n}, \mathbb{C})_\sigma$ on $\Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$ given by $(k, h) \cdot g = kgh^{-1}$. Note that $\Lambda^+ SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$ is connected but $\Lambda SO^+(1, \tilde{n})_\sigma$ has two connected components, since $\pi_1(SO^+(1, \tilde{n})) = \pi_1(SO(\tilde{n})) = \mathbb{Z}/2\mathbb{Z}$ when $\tilde{n} \geq 3$.

Similar to the case of the Birkhoff decomposition one should first consider the universal cover $Spin(1, \tilde{n}, \mathbb{C})$ of $SO^+(1, \tilde{n}, \mathbb{C})$, a two-fold covering. The subgroup $SO^+(1, \tilde{n})$ of $SO^+(1, \tilde{n}, \mathbb{C})$ has as preimage in $Spin(1, \tilde{n}, \mathbb{C})$ the group $Spin(1, \tilde{n})^0$ which is simply connected (For more details on the spin groups, see [45]). We have thus for the untwisted loop groups the "Iwasawa decomposition"

$$\Lambda Spin(1, \tilde{n}, \mathbb{C}) = \bigcup_{\tilde{\delta} \in \tilde{\Xi}} \Lambda Spin(1, \tilde{n})^0 \cdot \tilde{\delta} \cdot \Lambda^+ Spin(1, \tilde{n}, \mathbb{C}). \tag{93}$$

In the work of Kellersch [42], a detailed description of $\tilde{\Xi}$ was given. We will not need such details for this paper. We are primarily interested in a description of all open Iwasawa cells. Clearly, $\Lambda Spin(1, \tilde{n})^0 \cdot \Lambda^+ Spin(1, \tilde{n}, \mathbb{C})$ is open in $\Lambda Spin(1, \tilde{n}, \mathbb{C})$.

Theorem 6.5. *$\Lambda Spin(1, \tilde{n}, \mathbb{C})$ has exactly one open and dense Iwasawa cell with regard to $\Lambda Spin(1, \tilde{n})^0$.*

Proof. Let U denote a maximal compact subgroup of $Spin(1, \tilde{n}, \mathbb{C})$. Then U is simply connected, since $Spin(1, \tilde{n}, \mathbb{C})$ is simply connected. Now the claim follows from [42], Theorem 4.58. ☐

Let's now consider the case of twisted loop groups. Let σ denote the involution of $\mathfrak{so}(1, \tilde{n}, \mathbb{C})$ defining $\mathfrak{so}(1, 3) \oplus \mathfrak{so}(\tilde{n} - 3)$ inside $\mathfrak{so}(1, \tilde{n})$ and τ the anti-linear involution defining $\mathfrak{so}(1, \tilde{n})$ in $\mathfrak{so}(1, \tilde{n}, \mathbb{C})$. Finally, let θ denote the anti-linear involution defining $\mathfrak{so}(1 + \tilde{n})^* \cong \mathfrak{so}(1 + \tilde{n})$ in $\mathfrak{so}(1 + \tilde{n}, \mathbb{C})$.

All these involutions have extensions to involutions of $Spin(1+\tilde{n},\mathbb{C})$ and commute pairwise. We thus obtain

$$\Lambda Spin(1,\tilde{n},\mathbb{C})_\sigma = \bigcup_{\hat{\delta}\in\hat{\Xi}} \Lambda Spin(1,\tilde{n})^0_\sigma \cdot \hat{\delta} \cdot \Lambda^+ Spin(1,\tilde{n},\mathbb{C})_\sigma. \tag{94}$$

and via the natural projection

Theorem 6.6. *(Iwasawa decomposition theorem)*

$$\Lambda(SO^+(1,\tilde{n},\mathbb{C})_\sigma)^0 = \bigcup_{\delta\in\Delta} \Lambda SO^+(1,\tilde{n})^0_\sigma \cdot \delta \cdot \Lambda^+ SO^+(1,\tilde{n},\mathbb{C})_\sigma. \tag{95}$$

Moreover, the image of any open Iwasawa cell in $\Lambda Spin(1,\tilde{n},\mathbb{C})_\sigma$ is an open Iwasawa cell in $\Lambda(SO^+(1,\tilde{n},\mathbb{C})_\sigma)^0$.

Theorem 6.7. $\Lambda(SO^+(1,\tilde{n},\mathbb{C})_\sigma)^0$ *has exactly two open Iwasawa cells relative to* $\Lambda SO^+(1,\tilde{n})^0_\sigma$.

Proof. The argument preceding Proposition 4.5.2 in [17] shows that if $g = h\delta v_+$ is in an open Iwasawa cell, $(\tau(g))^{-1}g$ is in the big Birkhoff cell. Moreover, $(\tau(\delta))^{-1}\delta = q \in K^{\mathbb{C}} \cap U$, where $U = Fix\theta$ and $K^{\mathbb{C}} = Fix\tau \cong SO^+(1,3,\mathbb{C}) \times SO(\tilde{n}-3,\mathbb{C})$ in $SO^+(1,\tilde{n},\mathbb{C})$. Kellersch has shown in [42], Theorem 4.40, that q can be written in the form

$$q = \tau(u)^{-1}u, \text{ for some } u \in U. \tag{96}$$

(This u may not be fixed by σ).

In our setting $U = SO(1+\tilde{n})^*$. Every $u \in U$ can be written in the form $u = h \cdot \exp p$ with $\tau(h) = h$ $\tau(p) = -p$, since τ is a (real) involution of $U : A \mapsto \bar{A}$. Hence $q = (\tau(u))^{-1}u = \exp 2p$. In our case, $p \in LieU$ is of the form

$$p = \begin{pmatrix} 0 & ia^t \\ ia & 0 \end{pmatrix}, \; a \in \mathbb{R}^n.$$

A straightforward computation shows

$$\exp p = \begin{pmatrix} \cos||a|| & i\frac{\sin||a||}{||a||}a^t \\ i\frac{\sin||a||}{||a||}a & I + \frac{\cos||a||-1}{||a||^2}aa^t \end{pmatrix}. \tag{97}$$

We now return to the discussion of the equation

$$q = (\tau(u))^{-1}u = \exp(-\tau(p))\exp p = \exp 2p.$$

Since q is fixed by σ, we derive from (97) that at most the first three coefficients of $\frac{\sin 2||a||}{2||a||}2a^t$ do not vanish.

Case 1. $\sin 2||a|| \neq 0$. In this case we obtain that at most the first three coefficients of a do not vanish and $\exp p$ is actually fixed by σ. Replacing $u = h\exp p$ by $\exp p$ then shows that we can assume w.l.g. $\sigma(u) = u$.

Case 2. $\sin 2||a|| = 0$. We can also assume $\sin||a|| \neq 0$, otherwise $\exp p$ is fixed under τ and $q = id$ follows. In the present situation we derive from (96) that the matrix $\exp p$ is fixed under σ only if the matrix $P = \frac{\cos 2||a||-1}{4||a||^2}(2a)(2a)^t$ is of the form

$$\begin{pmatrix} A_0 & 0 \\ 0 & B_0 \end{pmatrix}$$

with A_0 a 3×3 matrix. We can assume $a \neq 0$. By our assumption $\sin 2||a|| = 0$ we know $\cos 2||a|| = \pm 1$. The case $\cos 2||a|| = 1$ yields $q = id$. In the case $\cos 2||a|| = -1$ we actually obtain that the matrix aa^t is a block matrix as above. Assume the $j-$th row, $a_j a^t$ does not vanish. Then $a_j \neq 0$ and a has at most the first three coefficients non-vanishing or the last $\tilde{n} - 3$ coefficients non-vanishing.

If the first three coefficients only can be non-vanishing, then $\sigma(p) = p$ and $u = \exp p$ is fixed by σ. In the remaining case $\sigma(p) = -p$ and $\sigma(\exp 2p) = \exp 2p$. Since the row a of p is real , there exists some $k \in SO(\tilde{n}) \subset Fix\sigma \cap Fix\tau \cap U$ such that $(\tau(k))^{-1} qk = k^{-1} \exp 2pk = \exp 2(k^{-1}pk)$ is defined by $\hat{p}$ with representing vector

$$\hat{a} = (0, 0, 0, \pi, 0, \cdots, 0).^t$$

Then the matrix $q_0 = (\tau(k))^{-1} qk = \exp 2\hat{p}$ represents the same (open) Iwasawa cell as q did. Note this matrix is of the form

$$q_0 = \begin{pmatrix} -1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \tag{98}$$

Also note that q is in $\Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$, since it is of the form $\exp \hat{p}$. So far we have shown that at most two open Iwasawa cells exist in $\Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$, namely the ones associated with $\delta = id$ and $(\tau(\delta))^{-1}\delta = q_0$.

The discussion starting with equation (96) dealt with elements in U. In all but the last case we obtained $q = id$ or $q = (\tau(k))^{-1}k$ with $k \in \Lambda^+ SO^+(1, \tilde{n}, \mathbb{C})_\sigma$. For the last matrix, q_0, we only have a representation of the form $q_0 = (\tau(\exp \hat{p}))^{-1} \exp \hat{p}$ with $\exp \hat{p}$ in the untwisted loop group $\Lambda SO^+(1, \tilde{n}, \mathbb{C})$. Several possibilities exist, a priori:

(a). q_0 cannot be written in the form $q_0 = \tau(b)^{-1}b$ for any $b \in \Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$.

(b). q_0 can be written in the form $q_0 = \tau(b)^{-1}b$ for some $b \in \Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$, but actually $b \in \Lambda^+ SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$.

(c). q_0 can be written in the form $q_0 = \tau(b)^{-1}b$ for some $b \in \Lambda SO^+(1, \tilde{n}, \mathbb{C})_\sigma$, but no such b is in $\Lambda^+ SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$.

Let's put $\hat{p}^* = D(\lambda)\hat{p}D(\lambda)^{-1}$ with

$$D(\lambda) = \mathrm{diag}\{\sqrt{\lambda}, 1, 1, 1, \sqrt{\lambda}^{-1}, 1, \cdots, 1\},$$

then $\hat{p}^* \in \Lambda \mathfrak{so}(1, \tilde{n}, \mathbb{C})_\sigma$ and $\exp 2\hat{p}^* = \exp 2\hat{p}$. Thus the possibility (a) is ruled out. (Note that conjugation with $D(\lambda)$ is a well defined operation on $\Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$.)

The possibility (b) can be ruled out, since $\tau(v_+)^{-1}v_+ = q_0$, equivalently $(\tau v_+)q_0 = v_+$, does not have a solution in $\Lambda SO^+(1, \tilde{n}, \mathbb{C})^0_\sigma$. It therefore only remains to show that the Iwasawa cell $\Lambda SO^+(1, \tilde{n})^0_\sigma \cdot \delta \cdot \Lambda^+ SO^+(1, \tilde{n}, \mathbb{C})_\sigma$ is open. For this it suffices to prove $\Lambda \mathfrak{so}(1, \tilde{n})_\sigma + \delta^{-1}\Lambda^+ \mathfrak{so}(1, \tilde{n}, \mathbb{C})_\sigma \delta = \Lambda \mathfrak{so}(1, \tilde{n}, \mathbb{C})_\sigma$, but this follows from a straightforward computation. $\square$

7 Appendix B: inner symmetric spaces

Proof of Lemma 4.20.

Let $G^{\mathbb{C}}$ be a simply connected complex Lie group and σ, θ, τ be as in Section 4.4. Since G/K is inner symmetric, there exists some $h \in G$ such that

$$\sigma(g) = hgh^{-1} = Adhg, \text{ for all } g \in G.$$

Hence $\sigma^2 = id$, equivalently, $(Adh)^2 = Id$ holds. Note that Adh is a semi-simple linear transformation of $\mathfrak{g}^{\mathbb{C}}$. Therefore, by definition, h is a semi-simple element of $G^{\mathbb{C}}$. Now we need (see e.g. Theorem 3.3.9 in [53])

Theorem 7.1. *A connected complex Lie group coinciding with its commutator group and having a faithful linear representation admits a unique algebraic structure.*

This implies (see e.g. Corollary 1 of [53] and p.113)

Corollary 7.2. *Every semi-simple element of $G^{\mathbb{C}}$ is contained in an "algebraic torus" $\cong (\mathbb{C}^*)^n$ of G.*

Then we have that $h \in H \subset Cent(h)^0$ where H is a maximal algebraic torus in $G^{\mathbb{C}}$ ([53], p. 214 last line). Let $P = Cent_G(h)^0$. Then P is a connected complex Lie group and it is invariant under θ. To show this, we observe first

$$php^{-1} = h \;\Rightarrow\; \theta(p)\theta(h)(\theta(p))^{-1} = \theta(h), \text{ for all } p \in P. \tag{99}$$

Since $\sigma \circ \theta = \theta \circ \sigma$, we obtain

$$h\theta(x)h^{-1} = \theta(hxh^{-1}) = \theta(h)\theta(x)(\theta(h))^{-1}, \text{ for all } x \in G^{\mathbb{C}}.$$

This is equivalent to

$$(h^{-1}\theta(h))\theta(x) = \theta(x)(h^{-1}\theta(h)), \text{ for all } x \in G^{\mathbb{C}},$$

i.e.,

$$(h^{-1}\theta(h)) \in Center(G).$$

So there exists $c \in Center(G)$ such that $\theta(h) = hc$. Hence (99) implies for all $p \in P$

$$\theta(p) = \theta(h)\theta(p)(\theta(h))^{-1} = (hc)\theta(p)(hc)^{-1} = h\theta(p)h^{-1}.$$

Thus P is a complex Lie group which is invariant under θ. Moreover, P is an algebraic subgroup of $G^{\mathbb{C}}$. Concerning the Lie algebra of P , we have

$$LieP = LieP \cap LieU + LieP \cap (LieU)^{\perp}.$$

Since $(LieU)^{\perp} = iLieU$, we see that the second summand is $LieP \cap (LieU)^{\perp} = i(LieP \cap LieU)$. Therefore $LieP \cap LieU$ is a maximal compact subalgebra of $LieP$. Let $\mathfrak{a} \subset LieP \cap LieU$ be a maximal abelian subalgebra , and $\mathfrak{c} = \mathfrak{a} + i\mathfrak{a}$. Then $\mathfrak{c}$ is a maximal algebraic torus in $LieP$. $\mathfrak{c}$ is conjugate to the maximal algebraic torus of $LieH$. Therefore $\mathfrak{a}$ is a maximal algebraic torus in $(LieP \cap LieU) \subset LieU$. Obviously $\mathfrak{a}$ is pointwise fixed by σ. Hence, σ is an automorphism of U fixing a maximal algebraic torus point-wise. From Proposition 53, Chap. IX of [36] it now follows that $U/(U \cap K^{\mathbb{C}})$ is an inner symmetric space. (Also see [53], Chap. 4, §4.) □

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